Structures of opposition in formal concept analysis, rough set theory, possibility theory and qualitative fuzzy integrals

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Structures of opposition as a unifying framework

- **Introduction**: squares and hexagons
- From the square to the cube
- The generality of the cube
- Example 1. *Boolean* Possibility theory
- Example 2. Formal Concept Analysis
- Example 3. Rough Set Theory
- *Graded* structures
- Example 1 continued. Possibility theory
- Example 4. Qualitative fuzzy integrals
- Example 5. Argumentation
- Example 6. Analogical proportion
Introduction 2. AI, squares and hexagons

- **Square of opposition** largely unknown in the information management sciences (AI, OR, ...)

- Also unknown:
  
  Robert Blanché’s last book (1966)
  
  *Structures Intellectuelles. Essai sur l’Organization Systématique des Concepts* 
  
  *from the square to the hexagon*

- Rare exceptions
  
  in relation with the use of modal logic systems

- Recent discovery of the relevance of squares, cube, hexagons, octagons in AI and related areas
The square of oppositions

A: $\forall x \, p(x)$  Contraries  E: $\forall x \, \neg p(x)$

I: $\exists x \, p(x)$  Sub-contraries  O: $\exists x \, \neg p(x)$

Affirm / Negation

$\{x \mid p(x)$ holds$} \neq \emptyset$  $\{x \mid \neg p(x)$ holds$} \neq \emptyset$

Another instance:

A: $\Box p$  E: $\Box \neg p$  I: $\Diamond p$  O: $\Diamond \neg p$

where $\Diamond p =_{def} \neg \Box \neg p$

(with $p \neq \bot, \top$)

Henri Prade (IRIT)  Structures of opposition  Graphs&Decisions’14,Luxemb.
**Constraints in the square**

- (a) \( A \) and \( O \) are the negation of each other, as well as \( E \) and \( I \)
- (b) \( A \) entails \( I \), and \( E \) entails \( O \)
- (c) \( A \) and \( E \) cannot be true together, but may be false together
- (d) \( I \) and \( O \) cannot be false together, but may be true together
The hexagon of opposition

Figure: Hexagon induced by a complete preorder (Robert Blanché, 1953)

\[ U = A \vee E \quad Y = I \wedge O \]

Three squares (A. Sesmat, 1951, P. Jacoby, 1950)

Renewal of interest initiated by Jean-Yves Béziau’s pioneering work
A deontic instance of the hexagon

[Diagram showing the deontic hexagon with the following terms: obligatory, regulated, forbidden, permitted, non obligatory, non regulated.]
**Why hexagons?**

Hexagon induced by a **tri-partition** \((A, B, C)\)

**Triangle** of 3 *mutually exclusive* situations
What about 4 mutually exclusive situations?

Example: A: >  B: <  C: =  D: incomparable
Example: tetrahedron of binary connectives

Similar to tetraisohedron
Luzeaux, Sallantin, and Dartnell ; Moretti
From square to cube

A: all $P$ are $Q$

E: all $P$ are $\overline{Q}$

I: at least a $P$ is a $Q$

O: at least a $P$ is a $\overline{Q}$
Cube of opposition (after De Morgan)

\[ \begin{align*}
a & : \text{all } \overline{P} \text{ are } \overline{Q} \\
A & : \text{all } P \text{ are } Q \\
i & : \text{at least a } \overline{P} \text{ is a } \overline{Q} \\
I & : \text{at least a } P \text{ is a } Q \\
E & : \text{all } P \text{ are } \overline{Q} \\
o & : \text{at least a } \overline{P} \text{ is a } Q \\
O & : \text{at least a } P \text{ is a } \overline{Q}
\end{align*} \]

6 squares ! 4 different structures ...
Piaget’s group of logical transformations

logical formula $\phi = f(p, q, r, \ldots)$

- identity $I(\phi) = \phi$
- negation $N(\phi) = \neg \phi$
- reciprocation $R(\phi) = f(\neg p, \neg q, \neg r, \ldots)$
- correlation $C(\phi) = \neg f(\neg p, \neg q, \neg r, \ldots)$

$N = RC$, $R = NC$, $C = NR$, et $I = NRC$

Klein’s group with 4 elements

at work in the two diagonal rectangles AaOo and Eeli
A Relation $R$ and a Subset $S$

binary relation $R \neq \emptyset$ on $X \times Y$ (one may have $Y = X$)

$xR = \{y \in Y | (x, y) \in R\}$

normalization assumption $\forall x \ xR \neq \emptyset$

we write $xRy$ for $(x, y) \in R$, and $\neg (xRy)$ for $(x, y) \notin R$

subset $S \subseteq Y$

It gives birth to the two subsets

$R(S) = \{x \in X | \exists s \in S, xRs\} = \{x \in X | S \cap xR \neq \emptyset\}$

$R(\overline{S}) = \{x \in X | \exists s \in \overline{S}, xRs\}$

and their complements

$\overline{R(S)} = \{x \in X | \forall s \in S, \neg (xRs)\}$

$\overline{R(\overline{S})} = \{x \in X | \forall s \in \overline{S}, \neg (xRs)\} = \{x \in X | xR \subseteq S\}$
A square of opposition, as the square of modalities

- \( R(\overline{S}) \) and \( R(\overline{S}) \) are complements, as \( \overline{R(S)} \) and \( R(S) \) assuming the X-normalization condition \( \forall x, xR \neq \emptyset \):
  - \( R(\overline{S}) \subseteq R(S) \), and \( \overline{R(S)} \subseteq R(\overline{S}) \)
  - \( R(\overline{S}) \cap R(S) = \emptyset \); one may have \( \overline{R(S)} \cup R(S) \neq Y \)
  - \( R(S) \cup R(\overline{S}) = X \); one may have \( R(S) \cap R(\overline{S}) \neq \emptyset \)
The complementary relation \( \bar{R} \)

\[ x \bar{R} y \text{ iff } \neg (xRy) \quad \bar{R} \neq \emptyset \text{ (i.e., } R \neq X \times Y) \]

assume the **X-normalization** of \( \bar{R} \), i.e. \( \forall x, \exists y \neg (xRy) \)

We get 4 other subsets of \( X \) from \( \bar{R} \)

\[
\bar{R}(\bar{S}) = \{ x \in X | \exists s \in \bar{S}, \neg (xRs) \} = \{ x \in X | S \cup xR \neq X \}
\]

\[
\bar{R}(S) = \{ x \in X | \exists s \in S, \neg (xRs) \}
\]

and their complements

\[
\bar{\bar{R}(S)} = \{ x \in X | \forall s \not\in S, xRs \}
\]

\[
\bar{R}(S) = \{ x \in X | \forall s \in S, xRs \} = \{ x \in X | S \subseteq xR \} \]
The 8 subsets can be organized into a cube of oppositions

Figure: Cube of oppositions induced by a relation \( R \) and a subset \( S \)
Figure: Top and bottom facets of the cube of oppositions
**Figure**: Hexagon associated with the front facet of the cube
Figure: Hexagon induced by the left-hand side square
Example 1: Boolean Possibility Theory-1

A Boolean possibility distribution $\pi : U \rightarrow \{0, 1\}$

- a subset $E$ of non impossible worlds: $E = \{u \mid \pi(u) = 1\}$;
- $E \neq \emptyset$; $E \neq U$

Another subset (event) $A$ is

- potentially possible if $A \cap E \neq \emptyset : \Pi(A) = 1$ (0 otherwise). if $x \in E$ then it is possibly in $A$.
- actually possible if $A \subseteq E : \Delta(A) = 1$ (0 otherwise). it is enough that $x \in A$ to be sure $x$ is possible.
- actually necessary if $E \subseteq A : N(A) = 1$ (0 otherwise). if $x \in E$ then it is surely in $A$.
- potentially necessary if $A \cup E \neq U : \nabla(A) = 1$ (0 otherwise). if $x \not\in E$ then it is possibly not in $A$

One has $\max(N(A), \Delta(A)) \leq \min(\Pi(A), \nabla(A))$: only 7 Boolean 4-tuples $(N(A), \Delta(A), \Pi(A), \nabla(A))$ out of 16.
Encoding the relative position of sets

There are 7 possible relative positions of $E$ and $A$:

<table>
<thead>
<tr>
<th>Position</th>
<th>$\Pi$</th>
<th>$\Delta$</th>
<th>$N$</th>
<th>$\nabla$</th>
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<td>$A = E$</td>
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<td>$A = E$</td>
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(Boolean) Possibility Theory-2

- (weak) possibility $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$
- (strong) necessity $\mathcal{N}(A \cap B) = \min(\mathcal{N}(A), \mathcal{N}(B))$
- (strong) possibility $\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$
- (weak) necessity $\nabla(A \cap B) = \max(\nabla(A), \nabla(B))$

Dualities $\mathcal{N}(A) = 1 - \Pi(\overline{A})$; $\Delta(A) = 1 - \nabla(\overline{A})$
**Possibilistic hexagon**

\[ N \lor \Delta \leq \Pi \land \nabla \]

*all lines express implications*
Possibilistic cube

$a$: $\Delta(A)$  
$b$: $\nabla(A)$  
$c$: $\Pi(A)$  
$d$: $\Pi(\overline{A})$  
$e$: $\Delta(\overline{A})$  
$f$: $\nabla(\overline{A})$  
$g$: $\Pi(\overline{A})$  
$h$: $\Pi(A)$

$A$: $N(A)$  
$E$: $N(\overline{A})$
Example 2: (Generalized) formal concept analysis

- relation $R \subseteq X \times Y = \text{Obj} \times \text{Prop}$ formal context
  - $R(x)$ the set of properties possessed by an object $x$, and
  - $R^{-1}(y)$ the set of objects having property $y$.

- $R^\cap(S) = \{x \in \text{Obj} | xR \cap S \neq \emptyset\} = \bigcup_{y \in S} R_y$
  - set of objects having at least a property in $S$

- $R^\cup(S) = \{x \in \text{Obj} | xR \subseteq S\} = \bigcap_{y \not\in S} R_y$
  - set of objects having no property outside $S$

- $R^\Delta(S) = \{x \in \text{Obj} | xR \supseteq S\} = \bigcap_{y \in S} R_y$
  - set of objects having all the properties in $S$

- $R^\land(S) = \{x \in \text{Obj} | xR \cup S \neq \text{Prop}\} = \bigcup_{y \not\in S} R_y$
  - set of objects to which at least a property outside $S$ is missing
**Formal concepts analysis**

A *formal concept* is a pair \((T, S)\) of extent and intent such that

\[
R^\Delta(T) = S \text{ and } R^{-1\Delta}(S) = T
\]

(B. Ganter and R. Wille)

This is equivalent to finding a pair of largest sets \((T, S)\) such that \(T \times S \subseteq R\)

at the basis of *data mining*
Example

Formal concepts: (\{g, h\}, \{2, 3, 4\}) ; (\{a, b, d, f\}, \{5, 6\}) ; (\{a, c, d\}, \{6, 7, 8\})

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The hexagon of formal concept analysis

\[ R^N(S) \cup R^\Delta(S) \subseteq R^\Pi(S) \cap R^\nabla(S) \]
Example 3: Rough sets

- Z. Pawlak, 1982
- \( R \) equivalence relation in \( X \times X \) here \( Y = X \)
  - \( R \) induces a partition of \( X \)
  - into equivalence classes of indiscernible elements

- The relation \( R \) is defined on the same domain \( R \subseteq X \times X \)
  - For any non empty subset \( S \subseteq X \), \( R \) induces a lower approximation \( L_R(S) = \{ x \in X | xR \subseteq S \} \)
  - upper approximation \( U_R(S) = \{ x \in X | xR \cap S \neq \emptyset \} \)
  - \( L_R(S) \subseteq S \subseteq U_R(S) \)
  - useful in machine learning
The front square

A square of opposition naturally arises from approximations

(A) \( \overline{R(S)} = L_R(S) \) is the lower approximation of \( S \) wrt the relation \( R \)

(I) \( R(S) = U_R(S) \) is the upper approximation of \( S \) wrt the relation \( R \)

(E) \( R(S) = L_R(\overline{S}) = \overline{U_R(S)} = E(S) \) is the exterior region of \( S \)

(O) \( R(\overline{S}) = U_R(\overline{S}) = L_R(S) \)
The back square

We need the sufficiency operator $[]$ and its dual $<>$

$$[[S]]_R = \{ x \in X | x^R \subseteq \overline{S} \}$$
$$<>S>_R = \{ x \in X | x^R \cap \overline{S} \neq \emptyset \} = \{ x \in X | S \cup x^R \neq X \}$$

(a) $\overline{R(S)} = [[S]] = \{ x : S \subseteq x^R \}$
the set of all $x$ which are in relation to all $y \in S$

(i) $\overline{R(S)} = <>S>>$ the set of objects which are not in relation to at least one object in $\overline{S}$

(e) $\overline{R(S)} = [[\overline{S}]]$

(o) $\overline{R(S)} = <>\overline{S}>>$
The cube from rough sets

Introduction  From square to cube  Generality of the cube  Examples  Graded structures  Examples 2  Conclusion / perspectives

Henri Prade (IRIT) Structures of opposition  Graphs & Decisions ’14, Luxemb.  33 / 53
Hexagon from the front square

3-partition of the universe $\text{Obj} = \text{L}(X) \cup \text{Bnd}(X) \cup \text{E}(X)$
Assume that a *grade in* \([0, 1]\) is associated with each vertex \(A, E, O,\) and \(I\) of the square of oppositions, namely \(\alpha, \epsilon, o,\) and \(i\) respectively. Then, the constraints

- (a) \(A\) and \(O\) are the negation of each other, as well as \(E\) and \(I\); can be encoded by \(\alpha = 1 - o; \epsilon = 1 - i\)
- (b) \(A\) entails \(I,\) and \(E\) entails \(O\); \(\alpha \leq i; \epsilon \leq o\)
- (c) \(A\) and \(E\) cannot be true together, but may be false together, can be encoded by \(\alpha + \epsilon \leq 1\)
- (d) \(I\) and \(O\) cannot be false together, but may be true together can be encoded by \(i + o \geq 1\)

easy to encode in a MV algebra!

Graded cube

\(\alpha, \iota, \epsilon, o, \) and \(\alpha', \iota', \epsilon', o'\) grades associated to A, I, E, O and a, i, e, o

Given an involutive negation \(n\), and a triangular norm \(*\)

The front facet and the back facet of the cube yield respectively

\((a)\) \(\alpha = n(o)\) and \(\epsilon = n(\iota)\)
\((b)\) \(\alpha * \epsilon = 0\)
\((c)\) \(n(\iota) * n(o) = 0\)
\((d)\) \(n(\alpha * n(\iota)) = 1\) and \(n(\epsilon * n(o)) = 1\)

\((a')\) \(\alpha = n(o')\) and \(\epsilon = n(\iota')\)
\((b')\) \(\alpha' * \epsilon' = 0\)
\((c')\) \(n(\iota') * n(o') = 0\)
\((d')\) \(n(\alpha' * n(\iota')) = 1\) and \(n(\epsilon' * n(o')) = 1\).

The constraints of the side facets translate into

- \(n(\alpha * n(\iota')) = 1\)  \(n(\alpha' * n(\iota)) = 1\)
- \(n(\epsilon' * n(o)) = 1\)  \(n(\epsilon * n(o')) = 1\)

Then the expected constraints hold in the top and bottom facets:

\(\alpha' * \epsilon = 0\)  \(\alpha * \epsilon' = 0\)
\(n(\iota') * n(o) = 0\)  \(n(\iota) * n(o') = 0\)
Example 1 continued: Possibility theory

i) (weak) possibility measure
\[ \Pi(A) = \max_{u \in A} \pi(u) \]
\[ \Pi(A \cup B) = \max(\Pi(A), \Pi(B)) \]

ii) dual (strong) necessity measure
\[ N(A) = \min_{u \not\in A} 1 - \pi(u) = 1 - \Pi(\overline{A}) \]
\[ N(A \cap B) = \min(N(A), N(B)) \]

iii) (strong) possibility measure
\[ \Delta(A) = \min_{u \in A} \pi(u) \]
\[ \Delta(A \cup B) = \min(\Delta(A), \Delta(B)) \]

iv) dual (weak) necessity measure
\[ \nabla(A) = \max_{u \not\in A} 1 - \pi(u) = 1 - \Delta(\overline{A}) \]
\[ \nabla(A \cap B) = \max(\nabla(A), \nabla(B)) \]
**Possibilistic hexagon**

\[ R^N(Y) \cup R^\Delta(Y) \subseteq R^\Pi(Y) \cap R^\nabla(Y) \]

**all lines express implications**
Example 4: Qualitative fuzzy integrals - 1

- **capacity** $\gamma : 2^C \rightarrow L$ s.t. $\gamma(\emptyset) = 0$, $\gamma(C) = 1$, $A \subseteq B \Rightarrow \gamma(A) \leq \gamma(B)$

- The **conjugate** $\gamma^c(A)$ of capacity $\gamma$ is a capacity defined by $\gamma^c(A) = 1 - \gamma(\overline{A})$, $\forall A \subseteq C$, where $\overline{A}$ is the complement of subset $A$

- Conjunctions and implications on a finite chain

\[
\begin{align*}
a \ast_D b &= a \land b \\
a \rightarrow_G b &= a \rightarrow_G b \\
\end{align*}
\]

\[
\begin{align*}
\text{Res} & \quad \text{Res} & \quad \text{Res} \\
S & \quad S & \quad S \\
\end{align*}
\]

\[
\begin{align*}
a \rightarrow_D b &= (1 - a) \lor b \\
a \ast_G b &= a \ast_G b \\
b \ast_G a &
\end{align*}
\]

\[
\begin{align*}
\text{Res} & \quad \text{Res} & \quad \text{Res} \\
S & \quad S & \quad S \\
\end{align*}
\]

\[
\begin{align*}
\bigwedge_{A \subseteq C} \gamma(A) \ast \bigwedge_{i \in A} f_i & \quad \text{and} \quad \bigvee_{A \subseteq C} \gamma_c(A) \rightarrow \bigvee_{i \in A} f_i \\
\end{align*}
\]

\[
\begin{align*}
\int_{\gamma}^* (f) &= \bigvee_{A \subseteq C} \gamma(A) \ast \bigwedge_{i \in A} f_i \\
\int_{\gamma}^\uparrow (f) &= \bigwedge_{A \subseteq C} \gamma_c(A) \rightarrow \bigvee_{i \in A} f_i \\
\end{align*}
\]

\[
\begin{align*}
\int_{\gamma}^* (f) &= 1 - \int_{\gamma}^\uparrow (1 - f) \\
\end{align*}
\]
Example 4: Qualitative fuzzy integrals - 2

**Sugeno integral** generalize $\bigvee_{i=1}^n \pi_i \land f_i$ and $\bigwedge_{i=1}^n (1 - \pi_i) \lor f_i$

$$\oint_{\gamma} (f) = \bigvee_{A \subseteq C} \gamma(A) \land \bigwedge_{i \in A} f_i = \bigwedge_{A \subseteq C} \gamma(\overline{A}) \lor \bigvee_{i \in A} f_i$$

**Soft integrals** generalize $\bigvee_{i=1}^n \pi_i \ast_G f_i$ and $\bigwedge_{i=1}^n \pi_i \rightarrow_G f_i$

$$\oint_{\gamma}^{\uparrow G} (f) = \bigwedge_{A \subseteq C} \gamma^c(A) \rightarrow_G \bigvee_{i \in A} f_i = \bigwedge_{A \subseteq C : \bigvee_{i \in A} f_i < \gamma^c(A)} \bigvee_{i \in A} f_i$$

$$\oint_{\gamma}^{\ast G} (f) = \bigvee_{A \subseteq C} \gamma(A) \ast_G \bigwedge_{i \in A} f_i = \bigvee_{A \subseteq C : \bigwedge_{i \in A} f_i > 1 - \gamma(A)} \gamma(A)$$

**Drastic integrals** generalize $\bigvee_{i=1}^n \pi_i \ast_{GC} f_i$ and $\bigwedge_{i=1}^n \pi_i \rightarrow_{GC} f_i$

$$\oint_{\gamma}^{\uparrow GC} (f) = \bigwedge_{A \subseteq C} \gamma^c(A) \rightarrow_{GC} \bigvee_{i \in A} f_i = \bigwedge_{A \subseteq C : \bigvee_{i \in A} f_i < 1 - \gamma^c(A)} (1 - \gamma^c(A))$$

$$\oint_{\gamma}^{\ast GC} (f) = \bigvee_{A \subseteq C} \gamma(A) \ast_{GC} \bigwedge_{i \in A} f_i = \bigvee_{A \subseteq C : \bigwedge_{i \in A} f_i > 1 - \gamma(A)} \gamma(A)$$
Example 4: Square and hexagon of oppositions

$\gamma$ a capacity

$\gamma_* = \min(\gamma, \gamma^c)$ is the *pessimistic* version of $\gamma$

$\gamma^* = \max(\gamma, \gamma^c)$ is the *optimistic* version

$\gamma^*$ is the conjugate of $\gamma_*$ and $\gamma_* \leq \gamma^*$

$A : S_{\gamma_*}(f)$  
$O : 1 - S_{\gamma_*}(f) = S_{\gamma^c}(1 - f) = S_{\gamma^*}(1 - f)$  
$I : S_{\gamma^*}(f)$  
$E : 1 - S_{\gamma^*}(f) = S_{\gamma^*}(1 - f)$

$S_{\gamma_*}(f) \leq S_{\gamma^*}(f)$ so $A$ entails $I$. Similarly $E$ entails $O$

If $\gamma$ is s.t. $\gamma \leq \gamma^c$ then $S_{\gamma}$ and $S_{\gamma}(1 - f)$ cannot be true together

$S_{\gamma_*}(f) \lor S_{\gamma^*}(1 - f)$
Example 4: Qualitative desintegrals

- **negative** scale: 0 is better than 1, but the scale of the global evaluation is increasing. The value \( f_i \) is all the greater as the evaluation is bad wrt criterion \( i \). The global score is all the greater as the resulting evaluation is better.

- **anti-capacity** \( \nu: 2^C \to L \) s.t. \( \nu(\emptyset) = 1, \nu(C) = 0, A \subseteq B \Rightarrow \nu(B) \leq \nu(A) \)

  The conjugate \( \nu^c \) of anti-capacity \( \nu \) is anti-capacity \( \nu^c(A) = 1 - \nu(A^c) \)

  A special case of anti-capacity; \( \Delta(A) = \land_{s \in A} \delta(s) \)

  where \( \delta \) is a possibility distribution such that \( \land_s \delta(s) = 0 \)

  The conjugate \( \nabla \) of \( \Delta \): \( \nabla(A) = 1 - \Delta(A^c) \)

  In *multiple criteria aggregation*, \( \delta(s) \) = *tolerance* level of criterion \( s \)

\[
\int_{\nu}^\downarrow (f) = \int_{1-\nu^c}^{\uparrow} (1 - f) \quad \text{and} \quad \int_{\nu}^{\star\downarrow} (f) = \int_{1-\nu^c}^{\star\uparrow} (1 - f)
\]

where \((\to, \star) \in \{(\to_D, \star_D), (\to_G, \star_G), (\to_{GC}, \star_{GC})\}\)
Example 4: Cube of oppositions

\[ a: S_{1-\gamma^c}(1 - f) \quad e: S_{1-\gamma^c}(f) \]
\[ A: S_{\gamma^*}(f) \quad E: S_{\gamma^*}(1 - f) \]
\[ o: S_{1-\gamma^c}(1 - f) \quad o: S_{1-\gamma^c}(f) \]
\[ i: S_{1-\gamma^c}(1 - f) \quad o: S_{1-\gamma^c}(f) \]

\[ A: S_{\gamma^*}(f) \quad E: S_{\gamma^*}(1 - f) \]

\[ i: S_{1-\gamma^c}(1 - f) \quad o: S_{1-\gamma^c}(f) \]

\[ I: S_{\gamma^*}(f) \quad O: S_{\gamma^*}(1 - f) \]
Conclusion

Structures of opposition (square, hexagon, cube, ...) are useful for analyzing, for extending, for bridging many information processing and AI theories

- for modeling epistemic uncertainty (*possibility th.*)
- for analyzing relations between objects and properties (*formal concepts*)
- for handling indiscernible objects (*rough sets*)
- in multi-criteria aggreg. (*qualit. fuzzy (des)integrals*)
- for *analogical proportion*-based reasoning
- for *argumentation*

as well as for *classical* and for *modal logic*

Interest for *graded structures*   *fuzzy relations*
References

- Didier Dubois, Henri Prade. From Blanché’s hexagonal organization of concepts to formal concept analysis and possibility theory. Logica Universalis, 6 (1-2), pp. 149-169, 2012.
Example 5: Abstract argumentation - 1

an abstract argumentation system: a pair \((A, R)\) (P. M. Dung, 1995)

- \(A\) is a set of arguments
- \(R \neq \emptyset\) a binary relation \(R \subseteq A \times A\)
- \((a, b) \in R\), or \(aRb\), means that \(a\) attacks \(b\)
- A subset \(S \subseteq A\) of arguments attacks an argument \(a\) if \(\exists s \in S\) and \(sRa\).
- A subset \(S \subseteq A\) of arguments attacks a subset \(S' \subseteq A\) if \(\exists s \in S\) and \(\exists s' \in S'\) and \(sRs'\)
- A subset \(S\) of arguments is conflict-free if \(\nexists (a, b) \in S \times S\) such that \(aRb\).
- \(R^+(S) = \{a \in A | S \text{ attacks } a\} = \{a \in A | S \cap Ra \neq \emptyset\}\)

the set of arguments attacking \(S\)
Abstract argumentation - 2

Key question: definition of subsets of acceptable arguments named extensions

A well-known form of acceptability is the notion of stable extension

- A conflict-free subset $S$ of arguments is a stable extension if and only if
  $$\forall a \notin S, \exists s \in S \text{ and } sRa.$$  

  A stable extension attacks all the arguments outside

  formal concept analysis // formal argumentation

- $R^\Delta(S) = \{y \in Y \mid S \subseteq R(y)\}$  
  $R^+(S) = \{a \in A \mid S \subseteq \overline{Ra}\}$

- A stable extension $S$  
  $$S = \overline{R^+(S)}$$  

  a formal concept $(R, S)$  
  $R^\Delta(T) = S$ and $R^\Delta(S) = T$

relation $\overline{R}$ (“does not attack”) plays the role of the formal context
**Forms of arguments**

Apothéloz points out the existence of four basic argumentative forms:

- “y is a reason for concluding x” (denoted C(x) : R(y))
- y is not a reason for concluding x” (C(x) : −R(y))
- y is a reason against concluding x” (−C(x) : R(y))
- y is not a reason against concluding x” (−C(x) : −R(y))

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**Possible argumentative relations**

linking a reason \( y \) to a conclusion \( x \)

\[
\begin{align*}
\text{U: } & \neg(\neg x, y) \\
\text{A: } & (x, y) \\
\text{E: } & \neg(x, y) \land \neg(\neg x, y) \\
\text{I: } & (x, y) \lor (\neg x, y) \\
\text{O: } & \neg(x, y) \\
\text{Y: } & (\neg x, y)
\end{align*}
\]
Example 6: Propositional view of the analogical proportion

- Analogical proportion “a is to b as c is to d”
- \[ a : b :: c : d \]
  \[ = ((a \land \neg b) \equiv (c \land \neg d)) \land ((\neg a \land b) \equiv (\neg c \land d)) \]
- a differs from b as c differs from d, and conversely, b differs from a as d differs from c
- true for the following 6 patterns:
  - 0 : 1 :: 0 : 1
  - 1 : 0 :: 1 : 0
  - 1 : 1 :: 0 : 0
  - 0 : 0 :: 1 : 1
  - 1 : 1 :: 1 : 1
  - 0 : 0 :: 0 : 0

- both a matter of a similarity and dissimilarity
A valid square of oppositions makes an analogical proportion true!

- A, E, I, O as the (Boolean-valued) vertices of a square of opposition
- A : E :: I : O form an analogical proportion when taken in this order since 0 : 0 :: 1 : 1, 0 : 1 :: 0 : 1 and 1 : 0 :: 1 : 0 are 3 of the 6 patterns that make an analogical proportion true
- 3 valid squares:

```
A 0 0 E 0 1 1
I 1 1 O 0 1 0
```

- What about the 3 other patterns 1 : 1 :: 0 : 0, 1 : 1 :: 1 : 1 and 0 : 0 :: 0 : 0?
They make ... a square of agreement
An analogical octagon

captures the construction of an analogical proportion from $R$ and $S$

$R \cap S : R \cap \overline{S} :: \overline{R} \cap S : \overline{R} \cap \overline{S}$

$R$ = “aerial” $\overline{R}$ = “aquatic” $S$ = “move on ground” $\overline{S}$ = “move above ground”

leads to state that “ants are to birds as crabs are to fishes”