

K-sorting with multiple ordinal criteria

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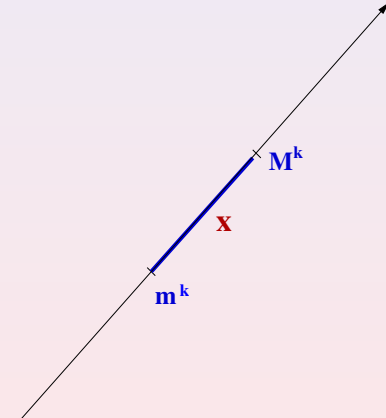
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MICS Algorithmic Decision Theory

K-Sorting on a single criteria

Category K is an interval $[m^k; M^k]$ on an ordinal measurement scale; x is a measured performance.

We may distinguish three sorting situations:



1. $x < m^k$ (and $x < M^k$)
The performance x is lower than category K ;
2. $x \geq m^k$ and $x < M^k$
The performance x belongs to category K ;
3. $(x \geq m^k \text{ and }) x \geq M^k$
The performance x is higher than category K .

If the relation $<$ is the dual of \geq , it will be sufficient to check that $x \geq m_k$ as well as $x \not\geq M_k$ are true for x to be a member of K .

Notations

- $A = \{x, y, z, \dots\}$ is a finite set of objects to be sorted.
- $F = \{1, \dots, n\}$ is a finite and coherent family of performance criteria.
- For each criterion i in F , the objects are evaluated on a real performance scale $[0; M_i]$, supporting an indifference threshold q_i and a preference threshold p_i such that $0 \leq q_i < p_i \leq M_i$.
- The performance of object x on criterion i is denoted x_i .
- Each criterion i in F carries a rational significance w_i such that $0 < w_i < 1.0$ and $\sum_{i \in F} w_i = 1.0$.

Performing marginally at least as good as

Each criterion i is characterising a double threshold order \geq_i on A in the following way:

$$r(x \geq_i y) = \begin{cases} +1 & \text{if } x_i + q_i \geq y_i \\ -1 & \text{if } x_i + p_i \leq y_i \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

- $+1$ signifies x is performing at least as good as y on criterion i ,
- -1 signifies that x is not performing at least as good as y on criterion i .
- 0 signifies that it is unclear whether, on criterion i , x is performing at least as good as y .

Performing globally *at least as good as*

Each criterion i contributes the significance w_i of his “*at least as good as*” characterisation $r(\geq_i)$ to the global characterisation $r(\geq)$ in the following way:

$$r(x \geq y) = \sum_{i \in F} [w_i \cdot r(x \geq_i y)] \quad (2)$$

- $r > 0$ signifies x is *globally performing at least as good as* y ,
- $r < 0$ signifies that x is *not globally performing at least as good as* y ,
- $r = 0$ signifies that it is *unclear* whether x is globally performing at least as good as y .

First result

Let $m^k = (m_1^k, m_2^k, \dots, m_p^k)$ denote the **lower limits** and $M^k = (M_1^k, M_2^k, \dots, M_p^k)$ the corresponding **upper limits** of category K on the criteria.

Proposition

That object x belongs to category K may be characterised as follows:

$$r(x \in K) = \min (r(x \geq m^k), r(x \not\geq M^k))$$

Performing marginally and globally *less than*

Each criterion i is characterising a double threshold order \ll_i (*less than*) on A in the following way:

$$r(x \ll_i y) = \begin{cases} +1 & \text{if } x_i + p_i \leq y_i \\ -1 & \text{if } x_i + q_i \geq y_i \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

And, the *global less than* relation (\ll) is defined as follows:

$$r(x \ll y) = \sum_{i \in F} [w_i \cdot r(x \ll_i y)] \quad (4)$$

Proposition

The global “*less than*” relation \ll is the **dual** ($\not\geq$) of the global “*at least as good as*” relation \geq .

Difference with Electre Tri

Roy introduced the concept of **veto threshold** v_i ($p_i < v_i \leq M_i + \epsilon$) to characterise the observation of *seriously less performing situations* on the family of criteria. This leads to a single threshold order, denoted \lll_i which characterises seriously less performing situations as follows:

$$r(x \lll_i y) = \begin{cases} +1 & \text{if } x_i + v_i \leq y_i \\ -1 & \text{otherwise} \end{cases} \quad (5)$$

And a global veto situation $x \lll y$ is characterised as:

$$r(x \lll y) = r\left(\bigvee_{i \in F} (x \lll_i y)\right) = \max_{i \in F} [r(x \lll_i y)] \quad (6)$$

The classic outranking relation

An object x *outranks* an object y , denoted $x \succcurlyeq y$, when:

1. a *significant majority* of criteria validates the fact that x is performing at least as good as s , i.e. $(x \succcurlyeq y)$.
2. And, there is *no veto* raised against this claim, i.e. $(x \not\llcurlyeq y)$.

The corresponding characteristic gives:

$$\begin{aligned} r(x \succcurlyeq y) &= r[(x \succcurlyeq y) \wedge (x \not\llcurlyeq y)] \\ &= \min [r(x \succcurlyeq y), -r(x \llcurlyeq y)] \end{aligned}$$

Difference with Electre Tri - continue

Proposition (Pirlot & Bouyssou 2009)

Let \succcurlyeq be the classic outranking relation.

- The asymmetric part \succ of the \succcurlyeq , i.e. $(x \succ y)$ and $(y \not\succeq x)$, is in general *not identical* to its codual relation $\not\succeq$.
- The *absence* of any *veto* situation is sufficient and necessary for making $\succ = \not\succeq$.

Corollary

In case no vetoes are observed, our approach gives similar results when compared with the Electre Tri method.

Marginal *seriously better* or *worse performing* situations

We redefine a single threshold order, denoted \lllcurlyeq_i which represents *seriously less performing* situations as follows:

$$r(x \lllcurlyeq_i y) = \begin{cases} +1 & \text{if } x_i + v_i \leq y_i \\ -1 & \text{if } x_i - v_i \geq y_i \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

And a corresponding dual *seriously better performing* situation \gggcurlyeq_i characterised as:

$$r(x \gggcurlyeq_i y) = \begin{cases} +1 & \text{if } x_i - v_i \geq y_i \\ -1 & \text{if } x_i + v_i \leq y_i \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Global *seriously better* or *worse performing* situations

A global *veto*, or *counter-veto* situation is now defines as follows:

$$r(x \lllcurlyeq y) = \bigoplus_{i \in F} r(x \lllcurlyeq_i y) \quad (9)$$

$$r(x \gggcurlyeq y) = \bigoplus_{i \in F} r(x \gggcurlyeq_i y) \quad (10)$$

where \bigoplus represents the epistemic polarising (Bisdorff 1997) or symmetric maximum (Grabisch et al. 2009) operator:

$$r \bigoplus r' = \begin{cases} \max(r, r') & \text{if } r \geq 0 \wedge r' \geq 0, \\ \min(r, r') & \text{if } r \leq 0 \wedge r' \leq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Characterising veto and counter-veto situations

1. $r(x \lll y) = 1$ iff there exists a criterion i such that $r(x \lll_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \ggg_j y) = 1$.
2. Conversely, $r(x \ggg y) = 1$ iff there exists a criterion i such that $r(x \ggg_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \lll_j y) = 1$.
3. $r(x \ggg y) = 0$ if either we observe no very large performance differences or we observe at the same time, both a very large positive and a very large negative performance difference.

Lemma

$r(\lll)^{-1}$ is identical to $r(\ggg)$.

Polarising the global “at least as good as” characteristic

The bipolar-valued characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = \begin{cases} 0, & \text{if } [\exists i \in F : r(x \lll_i y)] \wedge [\exists j \in F : r(x \ggg_j y)] \\ [r(x \geq y) \otimes -r(x \lll y)] & , \text{ otherwise.} \end{cases}$$

And in particular,

- $r(x \succsim y) = r(x \geq y)$ if no very large positive or negative performance differences are observed,
- $r(x \succsim y) = 1$ if $r(x \geq y) \geq 0$ and $r(x \ggg y) = 1$,
- $r(x \succsim y) = -1$ if $r(x \geq y) \leq 0$ and $r(x \lll y) = 1$,

The bipolar outranking relation \succsim

From an epistemic point of view, we say that:

1. **object x outranks object y** , denoted $(x \succsim y)$, if
 - 1.1 a **significant majority of criteria validates** a global outranking situation between x and y , and
 - 1.2 **no serious counter-performance** is observed on a discordant criterion,
2. **object x does not outrank object y** , denoted $(x \not\succsim y)$, if
 - 2.1 a **significant majority of criteria invalidates** a global outranking situation between x and y , and
 - 2.2 **no seriously better performing situation** is observed on a concordant criterion.

K-sorting with bipolar outrankings

Proposition

The dual ($\not\succsim$) of the bipolar outranking relation \succsim is identical to the strict converse outranking \succcurlyeq relation.

Proof:

$$\begin{aligned} r(x \not\succsim y) &= -r(x \succsim y) = -[r(x \geq y) \otimes -r(x \lll y)] \\ &= [-r(x \geq y) \otimes r(x \lll y)] \\ &= [r(x \not\geq y) \otimes -r(x \ggg y)] \\ &= [r(x < y) \otimes r(x \ggg y)] = r(x \succcurlyeq y). \end{aligned}$$

Corollary

The bipolar characteristic of y belonging to category K may be assessed as follows:

$$r(x \in K) = \min (r(x \succsim m^k), r(x \not\succsim M^k))$$

Properties of K-Sorting result

The multicriteria K-Sorting algorithm

- Input:** a set X of n objects with a performance table on a family of p criteria and a set \mathcal{C} of k empty categories K with lower and upper limits.
- For each object $x \in X$ and each category $K \in \mathcal{C}$**
 - $r(x \in K) \leftarrow \min(r(x \succ m^k), r(x \not\prec M^k))$
 - if $r(x \in K) \geq 0$:
add x to category K
- Output:** \mathcal{C}

Comment

- The complexity of the K-Sorting algorithm is linear: $\mathcal{O}(nkp)$.
- In case, \mathcal{C} represents p partitions of the criteria measurement scales, i.e. the upper limits of the preceding category correspond to the lower limits of the succeeding ones, there is a potential for reducing the complexity even more.

- Coherence:** Each object is always sorted into a possibly empty subset of adjacent categories.
- Weak Unicity:** In case of non overlapping categories and the absence of indeterminate bipolar outrankings, i.e. $r \neq 0$, every object is sorted into at most one category;
- Unicity:** If the categories represent a discriminated partition of the measurement scales on each criterion and $r \neq 0$, then every object is sorted into exactly one category;
- Independance:** The sorting result for object x , is independent of the other object's sorting results.
- Monotonicity:** If $r(x \succ y) = 1$, then x is sorted into a category which is at least as high ranked as the category into which is sorted object y .
- Stability:** If a category is dropped from \mathcal{C} , the content of the remaining categories will not change thereafter.

Some European universities

THE WORLD UNIVERSITY RANKINGS
 Times Higher Education
 POWERED BY THOMSON REUTERS

THE RANKINGS HOME TOP 200 ANALYSIS BY REGION BY SUBJECT SUBSCR

TOP EUROPEAN UNIVERSITIES 2010

REGION RANK	INSTITUTION	COUNTRY / REGION	OVERALL SCORE <small>change</small>
1	University of Cambridge	United Kingdom	91.2
1	University of Oxford	United Kingdom	91.2
3	Imperial College London	United Kingdom	90.6
4	Swiss Federal Institute of Technology Zurich	Switzerland	83.4

Edit the objects to sort

	active	id	or...	name
1	<input checked="" type="checkbox"/>	UM-UK		University of Manchester
2	<input checked="" type="checkbox"/>	RHL-...		Royal Holloway, University of London
3	<input checked="" type="checkbox"/>	LU-S		Lund University Sweden
4	<input checked="" type="checkbox"/>	UZ-CH		University of Zurich Switzerland
5	<input checked="" type="checkbox"/>	USth-...		University of Southampton
6	<input checked="" type="checkbox"/>	UCD-IR		University College Dublin
7	<input checked="" type="checkbox"/>	UB-CH		University of Basel
8	<input checked="" type="checkbox"/>	ENS...		Ecole Normale Superieure de Lyon
9	<input checked="" type="checkbox"/>	TUM-...		Technical University of Munich
10	<input checked="" type="checkbox"/>	UH-FI		University of Helsinki, Finland
11	<input checked="" type="checkbox"/>	UST...		University of St. Andrews
12	<input checked="" type="checkbox"/>	EUT-NL		Eindhoven University of Technology
13	<input checked="" type="checkbox"/>	UG-CH		University of Geneva
14	<input checked="" type="checkbox"/>	KUL-BE		Catholic University of Leuven, Belgium

THE evaluation criteria

The performances per university

Tune the sorting criterion

ac...	id	name	weight	direction	minimum	maximum
<input checked="" type="checkbox"/>	c-T	Teaching	3	max	0	100
<input checked="" type="checkbox"/>	c_I	International Mix	1	max	0	100
<input checked="" type="checkbox"/>	c-Ind	Industry income	1	max	0	100
<input checked="" type="checkbox"/>	c_R	Research	1	max	0	100
<input checked="" type="checkbox"/>	c_C	Citations	3	max	0	100

Criteria Weights Piechart

Edit the criterion discrimination thresholds

id	type	constant	proportion	percentile	description	cc	
1	th_c-T_ind	ind	1.0	0.025	0.12	proportional indifference threshold	
2	th_c-T_pref	pref	2.5	0.05	0.25	proportional preference threshold	
3	th_c-T_veto	veto	50.0		0.99	constant uncompensable performance difference	

1. problem configuration **2. edit performances** 3. criteria tuning 4. categories tuning 5. view sorting results

id	name	description	id	criterion	performance
4	UZ-CH	University of Zurich Switzerland	1	ev_c-T_KUL-BE c-T	57.7
5	UStH-UK	University of Southampton	2	ev_c_I_KUL-BE c_I	29.6
6	UCD-IR	University College Dublin	3	ev_c-Ind_KUL-BE c-Ind	97.7
7	UB-CH	University of Basel	4	ev_c_R_KUL-BE c_R	62.9
8	ENSL-FR	Ecole Normale Supérieure de Lyon	5	ev_c_C_KUL-BE c_C	45.2
9	TUM-DE	Technical University of Munich			
10	UH-FI	University of Helsinki, Finland			
11	USTA-UK	University of St. Andrews			
12	EUT-NL	Eindhoven University of Technology			
13	UG-CH	University of Geneva			
14	KUL-BE	Catholic University of Leuven, Belgium			

criterion	name	minimum	maximum	direction	description	
1	c-Ind	Industry income	0	100	max	innovation

Six sorting categories: A (best) - F (worst)

1. problem configuration 2. edit performances 3. criteria tuning **4. categories tuning** 5. view sorting results

Select a sorting criteria

id	ac...	name	direct...	mini...	maxi...
1	c-T	Teaching	max	0	100
2	c_I	International Mix	max	0	100
3	c-Ind	Industry income	max	0	100
4	c_R	Research	max	0	100
5	c_C	Citations	max	0	100

Criterion category limits

id	category	[lower limit -	- upperlimit [
1	lim_c_R_F	very weak	0 30
2	lim_c_R_E	weak	30 50
3	lim_c_R_D	fair	50 65
4	lim_c_R_C	good	65 80
5	lim_c_R_B	very good	80 90
6	lim_c_R_A	excellent	90 120

1. problem configuration 2. edit performances 3. criteria tuning 4. categories tuning **5. view sorting results**

perCategory perObject allSortingSituations

View category contents

computeSortingResults showOutrankings

Category contents

1 **Sorting results in descending order**

Categories	Assorting
[> - A]	['ICL-UK', 'UC-UK', 'UO-UK']
[A - B]	['EP-FR', 'ETHZ-CH', 'UCD-IR', 'UCL-UK']
[B - C]	['ENSP-FR', 'EP-FR', 'KI-S', 'TUM-DE', 'UCD-IR', 'UE-UK']
[C - D]	['ENSL-FR', 'EP-FR', 'EPFL-CH', 'EUT-NL', 'KCL-UK', 'KUL-BE', 'LSE-UK', 'LU-S', 'RKU-DE', 'TCD-IR', 'TUM-DE', 'UB-CH', 'UB-UK', 'UCD-IR', 'UG-CH', 'UG-DE', 'UH-FI', 'UM-DE', 'UM-UK', 'USTA-UK', 'UStH-UK', 'UY-UK', 'UZ-CH']
[D - E]	['DU-UK', 'RHL-UK', 'UB-CH', 'US-UK', 'USTA-UK']
[E - F]	[]

Select the category

rank	id	name
1	A	excellent
2	B	very good
3	C	good
4	D	fair
5	E	weak
6	F	very weak

Objects in the selected category

object	credibility (%)	>= low limit (%)	< high limit (%)
1 ICL-UK: Imperial College London	100	100.00	100.00
2 UO-UK: University of Oxford	55.5559921264...	55.56	100.00
3 UC-UK: University of Cambridge	55.5559921264...	55.56	100.00

Concluding ...

perCategory | perObject | **allSortingSituations**

Select an object

id	name
1	UM-UK University of Manchester
2	RHL-UK Royal Holloway, University of London
3	LU-S Lund University Sweden
4	UZ-CH University of Zurich Switzerland
5	USTH-UK University of Southampton
6	UCD-IR University College Dublin
7	UB-CH University of Basel
8	ENSL-FR Ecole Normale Supérieure de Lyon
9	TUM-DE Technical University of Munich
10	UH-FI University of Helsinki, Finland
11	USTA-UK University of St. Andrews
12	EUT-NL Eindhoven University of Technology
13	UG-CH University of Geneva
14	KUL-BE Catholic University of Leuven, Belgium

Sorting situations wrt all the categories

showPairwiseComparison

id	category	credibility	>= low limit	< upper limit
1	sit_KUL-BE_A	A: excellent	-100	100.00
2	sit_KUL-BE_B	B: very good	-100	100.00
3	sit_KUL-BE_C	C: good	-55.555992126...	100.00
4	sit_KUL-BE_D	D: fair	11.11110019683...	11.11
5	sit_KUL-BE_E	E: weak	-11.11110019683...	-11.11
6	sit_KUL-BE_F	F: very weak	-100	-100.00

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Performances barchart with category limits

showPerformancesBarchart

google chart

1

Performances of alternative KUL-BE with the limits of category D (in % of the criteria scales)

Category	Performance (%)
c_Ind	95
c_T	55
c_C	45
c_I	30
c_R	65

- A new efficient K-sorting algorithm
- Bipolar extension of the classic outranking
- New Decision Deck software tool available