K-sorting with multiple ordinal criteria

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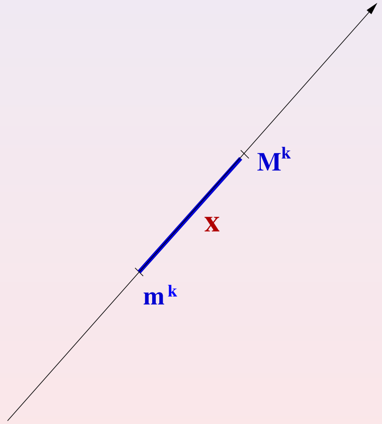
MICS Algorithmic Decision Theory

K-Sorting on a single criteria

Category K is an interval $[m^k; M^k[$ on an ordinal measurement scale; x is a measured performance.
We may distinguish three sorting situations:



1. $x < m^k$ (and $x < M^k$)
   The performance x is lower than category K;
2. $x \geqslant m^k$ and $x < M^k$
   The performance x belongs to category K;
3. ($x \geqslant m^k$ and) $x \geqslant M^k$
   The performance x is higher than category K.

If the relation $<$ is the dual of $\geqslant$, it will be sufficient to check that $x \geqslant m_k$ as well as $x \not\geqslant M_k$ are true for x to be a member of K.

Notations

- $A = \{x, y, z, ...\}$ is a finite set of objects to be sorted.
- $F = \{1, ..., n\}$ is a finite and coherent family of performance criteria.
- For each criterion i in F, the objects are evaluated on a real performance scale $[0; M_i]$,
  supporting an indifference threshold q_i
  and a preference threshold p_i such that $0 \leqslant q_i < p_i \leqslant M_i$.
- The performance of object x on criterion i is denoted x_i.
- Each criterion i in F carries a rational significance w_i such that $0 < w_i < 1.0$ and $\sum_{i \in F} w_i = 1.0$.

Performing marginally *at least as good as*

Each criterion i is characterising a double threshold order $\geqslant_i$ on A in the following way:

$$r(x \geqslant_i y) = \begin{cases} +1 & \text{if} \quad x_i + q_i \geqslant y_i \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

$+1$ signifies x is *performing at least as good as* y on criterion i,

-1 signifies that x is *not performing at least as good as* y on criterion i.

0 signifies that it is *unclear* whether, on criterion i, x is performing at least as good as y.

Performing globally *at least as good as*

Each criterion i contributes the significance w_i of his "*at least as good as*" characterisation $r(\geqslant_i)$ to the global characterisation $r(\geqslant)$ in the following way:

$$r(x \geqslant y) \;=\; \textstyle\sum_{i \in F} \left[\, w_i \cdot r(x \geqslant_i y) \,\right] \tag{2}$$

- $r > 0$ signifies x is *globally performing at least as good as* y,
- $r < 0$ signifies that x is *not globally performing at least as good as* y,
- $r = 0$ signifies that it is *unclear* whether x is globally performing at least as good as y.

Performing marginally and globally *less than*

Each criterion i is characterising a double threshold order $<_i$ (*less than*) on A in the following way:

$$r(x <_i y) \;=\; \begin{cases} +1 & \text{if} \quad x_i + p_i \leqslant y_i \\ -1 & \text{if} \quad x_i + q_i \geqslant y_i \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

And, the *global less than* relation ($<$) is defined as follows:

$$r(x < y) = \textstyle\sum_{i \in F} \left[w_i \cdot r(x <_i y) \right] \tag{4}$$

Proposition

The global "less than" relation $<$ is the dual ($\ngeqslant$) of the global "at least as good as" relation $\geqslant$.

First result

Let $m^k = (m^k_1, m^k_2, ..., m^k_p)$ denote the lower limits and $M^k = (M^k_1, M^k_2, ..., M^k_p)$ the corresponding upper limits of category K on the criteria.

Proposition

That object x belongs to category K may be characterised as follows:

$$r(x \in K) \;=\; \min\left(r(x \geqslant m^k), r(x \ngeqslant M^k) \right)$$

Difference with Electre Tri

Roy introduced the concept of veto threshold v_i ($p_i < v_i \leqslant M_i + \epsilon$) to characterise the observation of *seriously less performing situations* on the family of criteria. This leads to a single threshold order, denoted $\ll_i$ which characterises seriously less performing situations as follows:

$$r(x \ll_i y) \;=\; \begin{cases} +1 & \text{if} \quad x_i + v_i \leqslant y_i \\ -1 & \text{otherwise} \end{cases} \tag{5}$$

And a global veto situation $x \ll y$ is characterised as:

$$r(x \ll y) \;=\; r\Big(\bigvee_{i \in F} (x \ll_i y) \Big) = \max_{i \in F} \left[\, r(x \ll_i y) \,\right] \tag{6}$$

The classic outranking relation

An object x *outranks* an object y , denoted $x \succsim y$, when:

1. a *significant majority* of criteria validates the fact that x is performing at least as good as s, i.e. $(x \geqslant y)$.
2. And, there is *no veto* raised against this claim, i.e. $(x \not\ll y)$.

The corresponding characteristic gives:

$$
\begin{aligned}
r(x \succsim y) &= r\big[(x \geqslant y) \wedge (x \not\ll y)\big] \\
&= \min\big[\, r(x \geqslant y), -r(x \ll y) \big]
\end{aligned}
$$

Difference with Electre Tri - continue

Proposition (Pirlot & Bouyssou 2009)

Let $\succsim$ be the classic outranking relation.

- *The asymmetric part $\succnsim$ of the $\succsim$, i.e. $(x \succsim y)$ and $(y \not\succsim x)$, is in general not identical to its codual relation $\not\precsim$.*
- *The absence of any veto situation is sufficient and necessary for making $\succnsim = \not\precsim$.*

Corollary

In case no vetoes are observed, our approach gives similar results when compared with the Electre Tri method.

Marginal *seriously better* or *worse performing* situations

We redefine a single threshold order, denoted $\lll_i$ which represents *seriously less performing* situations as follows:

$$
r(x \lll_i y) = \begin{cases} +1 & \text{if} \quad x_i + v_i \leqslant y_i \\ -1 & \text{if} \quad x_i - v_i \geqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (7)
$$

And a corresponding dual *seriously better performing* situation $\ggg_i$ characterised as:

$$
r(x \ggg_i y) = \begin{cases} +1 & \text{if} \quad x_i - v_i \geqslant y_i \\ -1 & \text{if} \quad x_i + v_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (8)
$$

Global seriously *better* or *worse performing* situations

A global *veto*, or *counter-veto* situation is now defines as follows:

$$
\begin{aligned}
r(x \lll y) &= \circledvee_{i \in F}\, r(x \lll_i y) && (9) \\
r(x \ggg y) &= \circledvee_{i \in F}\, r(x \ggg_i y) && (10)
\end{aligned}
$$

where $\circledvee$ represents the epistemic polarising (Bisdorff 1997) or symmetric maximum (Grabisch et al. 2009) operator:

$$
r \circledvee r' = \begin{cases} \max(r, r') & \text{if} \quad r \geqslant 0 \wedge r' \geqslant 0, \\ \min(r, r') & \text{if} \quad r \leqslant 0 \wedge r' \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (11)
$$

Characterising veto and counter-veto situations

1. $r(x \lll y) = 1$ iff there exists a criterion i such that $r(x \lll_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \ggg_j y) = 1$.
2. Conversely, $r(x \ggg y) = 1$ iff there exists a criterion i such that $r(x \ggg_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \lll_j y) = 1$.
3. $r(x \ggg y) = 0$ if either we observe no very large performance differences or we observe at the same time, both a very large positive and a very large negative performance difference.

Lemma

$r(\not\lll)^{-1}$ is identical to $r(\ggg)$.

The bipolar outranking relation $\succsim$

From an epistemic point of view, we say that:

1. object x *outranks* object y, denoted $(x \succsim y)$, if
   1.1 a significant majority of criteria validates a global outranking situation between x and y, and
   1.2 no serious counter-performance is observed on a discordant criterion,
2. object x *does not outrank* object y, denoted $(x \not\succsim y)$, if
   2.1 a significant majority of criteria invalidates a global outranking situation between x and y, and
   2.2 no seriously better performing situation is observed on a concordant criterion.

Polarising the global "*at least as good as*" characteristic

The bipolar-valued characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = \begin{cases} 0, & \text{if } \big[\exists i \in F : r(x \lll_i y)\big] \wedge \big[\exists j \in F : r(x \ggg_j y)\big] \\ \big[\, r(x \geqslant y) \circledvee -r(x \lll y)\big] & \text{, otherwise.} \end{cases}$$

And in particular,

- $r(x \succsim y) = r(x \geqslant y)$ if no very large positive or negative performance differences are observed,
- $r(x \succsim y) = 1$ if $r(x \geqslant y) \geqslant 0$ and $r(x \ggg y) = 1$,
- $r(x \succsim y) = -1$ if $r(x \geqslant y) \leqslant 0$ and $r(x \lll y) = 1$,

K-sorting with bipolar outrankings

Proposition

The dual $(\not\succsim)$ of the bipolar outranking relation $\succsim$ is identical to the strict converse outranking $\not\precsim$ relation.

Proof:

$$\begin{aligned} r(x \not\succsim y) &= -r(x \succsim y) \;=\; -\big[r(x \geqslant y) \circledvee -r(x \lll y)\big] \\ &= \big[-r(x \geqslant y) \circledvee r(x \lll y)\big] \\ &= \big[r(x \not\geqslant y) \circledvee -r(x \ggg y)\big] \\ &= \big[r(x < y) \circledvee r(x \not\ggg y)\big] \;=\; r(x \not\precsim y). \end{aligned}$$

Corollary

The bipolar characteristic of y belonging to category K may be assessed as follows:

$$\boxed{r(x \in K) = \min\big(\, r(x \succsim m^k),\, r(x \not\succsim M^k)\,\big)}$$

The multicriteria K-Sorting algorithm

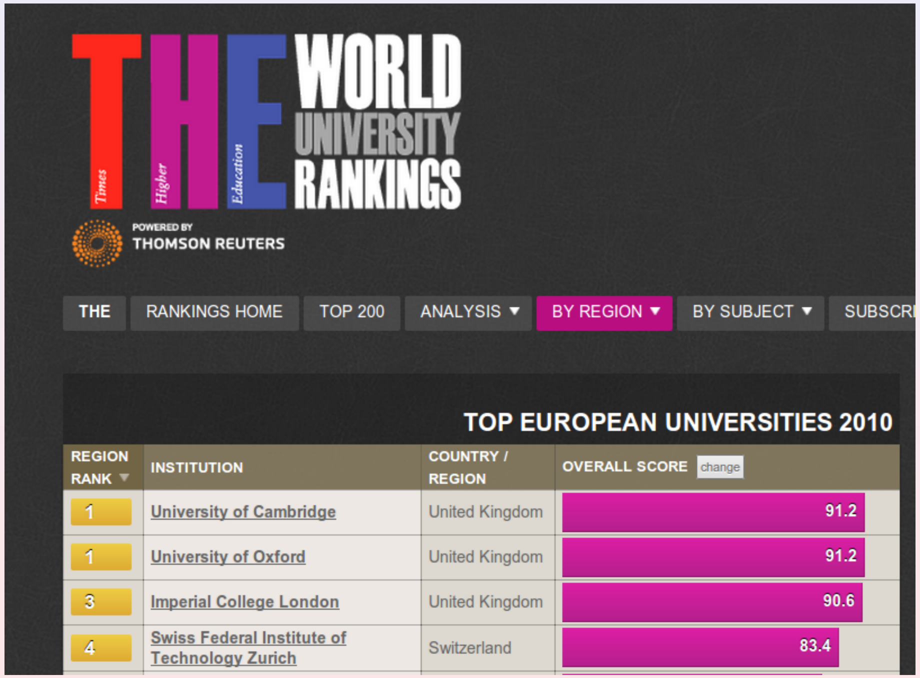
1. **Input**: a set X of n objects with a performance table on a family of p criteria and a set $\mathcal{C}$ of k empty categories K with lower and upper limits.
2. **For each** object $x \in X$ **and each** category $K \in \mathcal{C}$
   2.1 $r(x \in K) \quad \leftarrow \quad \min\big(r(x \succsim m^k), r(x \not\precsim M^k)\big)$
   2.2 if $r(x \in K) \geqslant 0$:
   add x to category K
3. **Output**: $\mathcal{C}$

Comment

1. *The complexity of the K-Sorting algorithm is linear: $\mathcal{O}(nkp)$.*
2. *In case, $\mathcal{C}$ represents p partitions of the criteria measurement scales, i.e. the upper limits of the preceding category correspond to the lower limits of the succeeding ones, there is a potential for reducing the complexity even more.*

Properties of K-Sorting result

1. *Coherence*: Each object is always sorted into a possibly empty subset of adjacent categories.
2. *Weak Unicity*: In case of non overlapping categories and the absence of indeterminate bipolar outrankings, i.e. $r \neq 0$, every object is sorted into at most one category;
3. *Unicity*: If the categories represent a discriminated partition of the measurement scales on each criterion and $r \neq 0$, then every object is sorted into exactly one category;
4. *Independance*: The sorting result for object x, is independent of the other object's sorting results.
5. *Monotonicity*: If $r(x \succsim y) = 1$, then x is sorted into a category which is at least as high ranked as the category into which is sorted object y.
6. *Stability*: If a category is dropped from $\mathcal{C}$, the content of the remaining categories will not change thereafter.



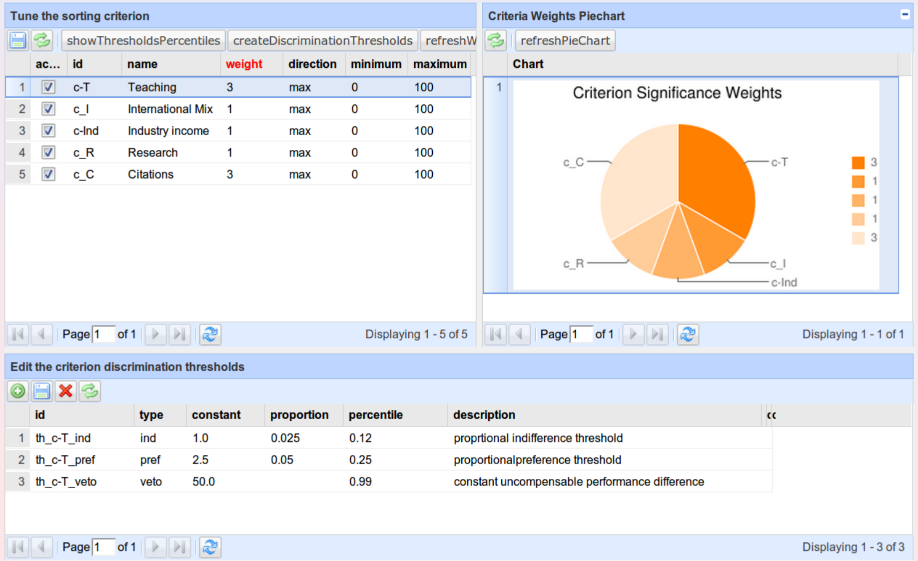
Some European universities

Edit the objects to sort

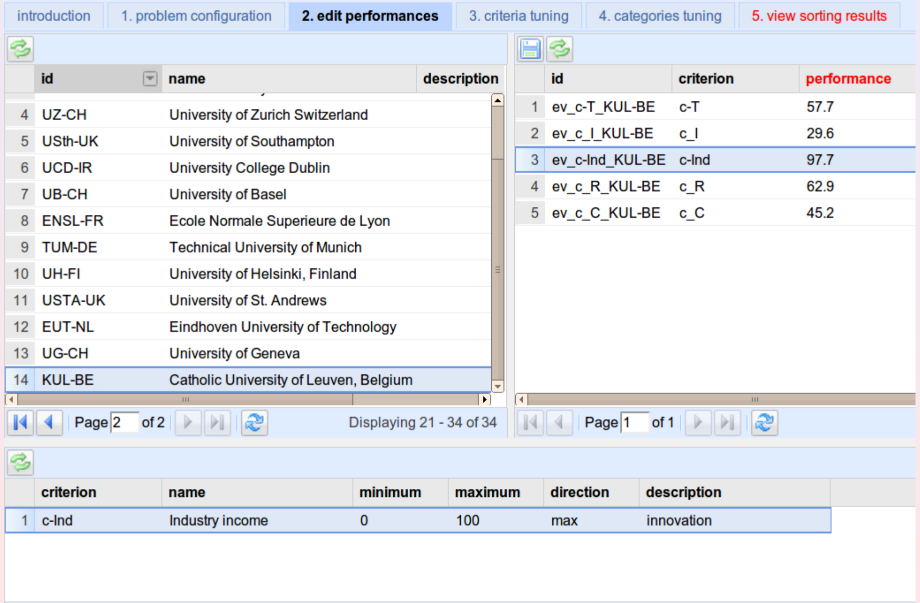
AboutObject

	active	id	or...	name
1	☑	UM-UK		University of Manchester
2	☑	RHL-...		Royal Holloway, University of London
3	☑	LU-S		Lund University Sweden
4	☑	UZ-CH		University of Zurich Switzerland
5	☑	USth-...		University of Southampton
6	☑	UCD-IR		University College Dublin
7	☑	UB-CH		University of Basel
8	☑	ENS...		Ecole Normale Superieure de Lyon
9	☑	TUM-...		Technical University of Munich
10	☑	UH-FI		University of Helsinki, Finland
11	☑	UST...		University of St. Andrews
12	☑	EUT-NL		Eindhoven University of Technology
13	☑	UG-CH		University of Geneva
14	☑	KUL-BE		Catholic University of Leuven, Belgium

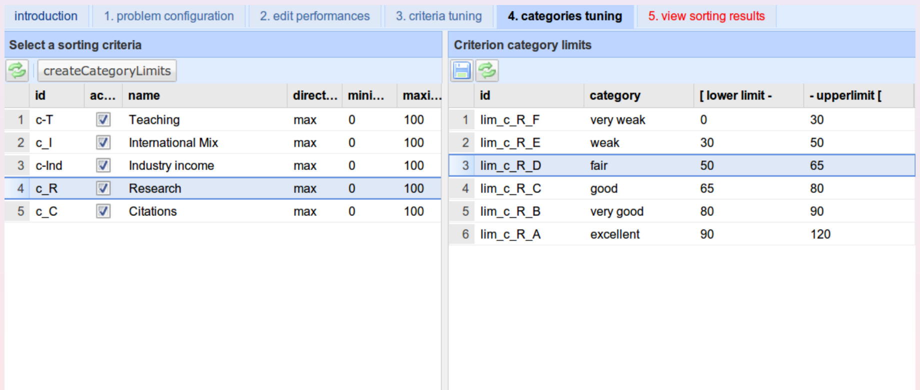
THE evaluation criteria

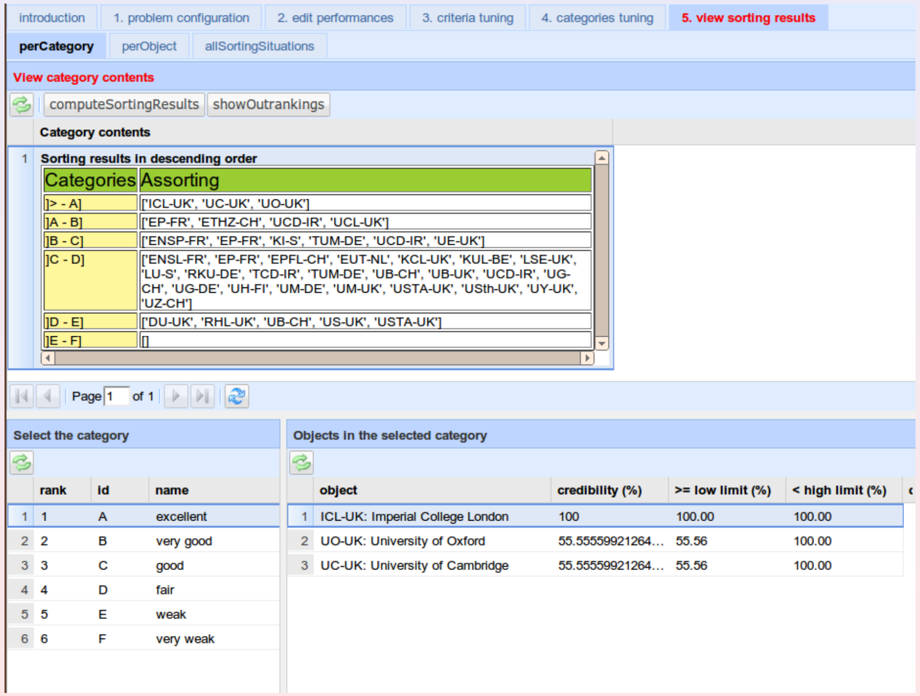


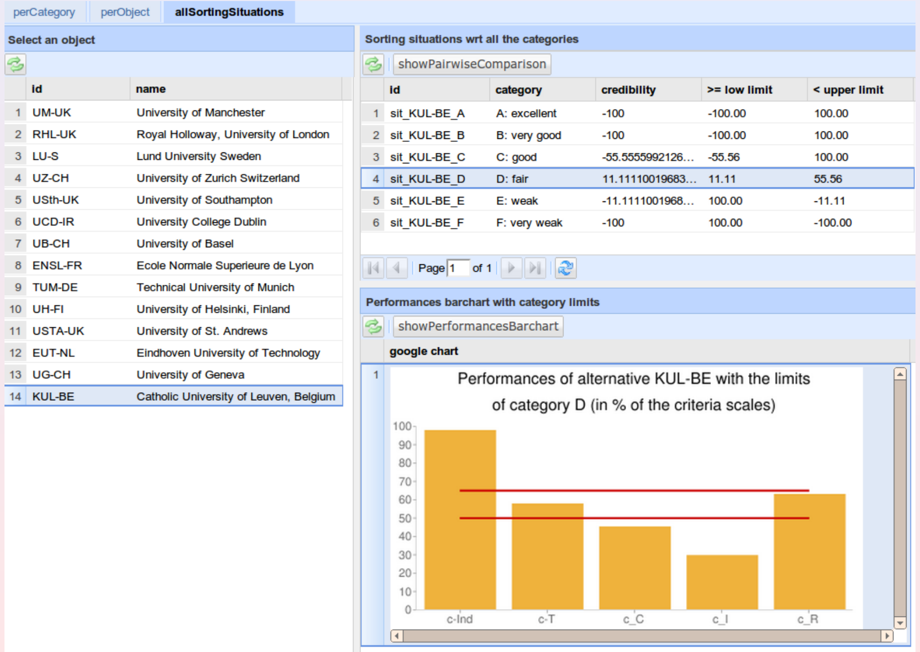
The performances per university



Six sorting categories: A (best) - F (worst)







Concluding ...

- A new efficient K-sorting algorithm
- Bipolar extension of the classic outranking
- New Decision Deck software tool available