On polarizing outranking relations with large performance differences

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Abstract

We introduce a bipolarly extended veto principle – a positive, as well as negative, large performance differences polarization – which allows us to extend the definition of the classical outranking relation in such a way that the identity between its asymmetric part and its codual relation is preserved.

Keywords: Multiple criteria decision aid; Outranking approach; Veto principle; Bipolar credibility valuation.

1 Introduction

In a recent conference, Pirlot and Bouyssou [1] reported that a strict (asymmetric) outranking relation defined similarly to the classical outranking [2] is formally not identical to its codual relation, meaning the negation of its converse. From value-based orderings, we are used to thinking that a decision alternative x is considered *strictly better than* a decision alternative y, when it is not true that y is *at least as good as* x. Consequently, we genuinely expect the '*strictly better than*' relation to be asymmetric. This will, however, only be the case if the corresponding '*at least as good as*' relation is complete, a fact which is usually not verified when dealing with classical outranking relations. This hiatus is problematic because the asymmetric part of an outranking relation is commonly identified as being its codual relation.

In this paper we explore this problem in the context of our bipolar-valued credibility calculus [3, 4, 5]. Logical characteristic functions will here denote the empirical validation, or not, of a preferential statement with the help of three states : *more true than false*, *more false than true*, or *logically indeterminate*. It is important to notice here, that in this bipolar setting, the logical negation operation can no longer be identified with standard set complementing. Contrary to classical logic, affirmation, as well as refutation of a preferential statement are here, both, based on explicit, not necessarily complementary, empirical arguments.

In the first section of this paper, we recall the basics of our bipolar-valued credibility calculus [3, 4, 6] in order to illustrate in the second section, following on from the work of Pirlot and Bouyssou [1], the unsound hiatus between the asymmetric part and the codual of the classical outranking concept [7]. In the third section we introduce a bipolarly extended large performance difference principle which allows us to adapt the definition of the classical outranking concept in such a way that the identity between its asymmetric part and its codual is indeed given. The final section is devoted to a numerical illustration.

2 The bipolar-valued credibility calculus

2.1 Well-formed statements

Let $\mathcal{P}$ represent a set of propositional ground statements p to which we may associate a rational number $r(p) \in [-1; +1]$, denoting the credibility degree of its potential truthfulness. If $r(p) = +1$, affirming statement p is considered to be "*certainly valid*". If $r(p) > 0$ (resp. $r(p) < 0$) , affirming p is considered to be "*more valid than invalid*" (resp. "*more invalid than valid*". If $r(p) = -1.0$, affirming statement p is considered to be "*certainly invalid*". Furthermore, for any propositional statements p and p', $r(p) > r(p')$ tells us that statement p is more likely to be valid than statement p', respectively that statement p' is more likely to be invalid than statement p. Note also that the limit case $r(p) = 0$ represents a situation of hesitation, where statement p appears to be *neither* valid *nor* invalid.

Let $\neg$, $\wedge$ and $\vee$ respectively denote the logical operators: negation, conjunction and disjunction. From a given finite set $\mathcal{P}$ of ground statements, grouping brackets and the basic logical operations, we may inductively generate the set $\mathcal{E}$ of all *well-formed finite statements* as follows:

$$\begin{aligned} \forall p \in \mathcal{P} \quad &: \quad p \in \mathcal{E}, \\ \forall x, y \in \mathcal{E} \quad &: \quad \neg x \mid (x) \mid x \vee y \mid x \wedge y \ \in \ \mathcal{E}. \end{aligned}$$

The negation operator $\neg$ has, as usual, a higher precedence in the interpretation of a well-formed statement, but usually we use brackets to control the application of a given logical operator so as to have unambiguous statement semantics.

2.2 Computing the credibility of a statement

The credibility denotation of the ground statements may be extended to all well-formed finite compound statements x, y in $\mathcal{E}$ as follows:

$$\begin{aligned} r(\neg x) &= -r(x), \\ r(x \vee y) &= \max(r(x), r(y)), \\ r(x \wedge y) &= \min(r(x), r(y)). \end{aligned}$$

In this extended bipolar-valued credibility calculus, the negation operator $-r()$ implements a strict anti-tonic bijection, with 0 acting as negational fixpoint [3]. Classical *min* and *max* operators capture the credibility of the logical conjunction and disjunction operations.

Knowing the credibility of any well-formed statement $p \in \mathcal{E}$, we are now able to characterize its supposed truthfulness.

2.3 Truthfulness of affirmative and refutative statements

Following on from the semantics of the bipolar-valued credibility, the truthfulness of a given well-formed statement $p \in \mathcal{E}$ may be expressed as follows: *affirming* p, respectively *refuting* p, is *valid* if $r(p) > -r(p)$, respectively $-r(p) > r(p)$; *affirming* p, respectively *refuting* p, is *invalid* if $r(p) < -r(p)$ (respectively $-r(p) < r(p)$; otherwise, when $r(p) = 0$, the situation is *indeterminate*. Hence, a statement is always *exclusively* either *valid*, either *invalid*, or neither valid nor invalid, that is *indeterminate*.

The *sign* of the credibility degree thus directly expresses whether the affirmation or the refutation of a well-formed statement is valid or not. And we may view the credibility $r(p)$ function as a three-valued logical characteristic function for any statement p taking value '*true*' if $r(p)$ is strictly positive (+), '*false*' if $r(p)$ is strictly negative ($-$), or '*indeterminate*' if $r(p) = 0$. In case no indeterminate value is given, the bipolar-valued credibility calculus will implement a classical Boolean algebra on a given set $\mathcal{E}$ of propositional statements.

We may now take a closer look at the particular set of ground propositional statements, obtained from pairwise comparisons of decision alternatives, which are relevant for our purposes.

3 The classical outranking concept

Let $A = \{x, y, z, \ldots\}$ be a finite set of potential decision alternatives and let $F = \{1, \ldots, n\}$ be a coherent, i.e. finite, exhaustive, cohesive and non redundant family of $n > 1$ criteria [2, see p. 103]. On each criterion $i \in F$, the alternatives are evaluated on a real valued performance scale, $[0; M_i]$, supporting coherent indifference, q_i, and preference, p_i, discrimination thresholds such that $0 \leqslant q_i < p_i \leqslant M_i$ [2]. The performance of alternative x on criterion i is denoted x_i. Without loss of generality we assume that the performance on all criteria is to be maximized.

3.1 Pairwise '*at least as good as*' statements

In order to characterize a marginal '*at least as good as*' situation [4, 8] between any two alternatives x and y, we associate each criterion i with a double threshold relation $\geqslant_i \subseteq A \times A$, whose bipolar-valued credibility function, $r(x \geqslant_i y)$ for all $(x, y) \in A^2$, is defined as follows:

$$r(x \geqslant_i y) = \begin{cases} 1 & \text{if} \quad x_i + q_i \;\geq\; y_i \\ -1 & \text{if} \quad x_i + p_i \;\leq\; y_i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Furthermore, we associate each criterion $i \in F$ with a *significance weight* $w_i \in \mathbb{Q}$, which represents the contribution of criterion i to the *overall warrant, or not,* of the '*at least as good as*' preference situation between all pairs of alternatives. Let W denote the list of relative significance weights associated with F such that:

$$W = [w_i : i \in F], \text{ with } 0 < w_i < 1 \text{ and } \sum_{i \in F} w_i = 1.$$

Following the tradition of the genuine outranking approach, we are going to compute the credibility of a global '*at least as good as*' statement over the whole family of criteria by additively balancing the weights of the criteria which support this statement against the weights of the criteria which do not do so.

Definition 3.1. Under the assumption that the family of criteria is indeed *coherent*, we can compute the bipolar-valued credibility r of the *overall* 'at least as good as' statement, denoted $\geqslant$ and aggregating all the marginal '*a*t least as good as' statements $\geqslant_i$, as follows[1]:

$$r(x \geqslant y) \quad := \quad \sum_{i \in F} \big[w_i \cdot r(x \geqslant_i y) \big], \tag{2}$$

[1] With $\frac{r+1}{2}$ we obtain the classical concordance index as used in the ELECTRE methods [2].

We call '*weakly complete*' a relation R on A characterized by a bipolar credibility function $r(\mathrm{R})$ such that: $\max[r(x\,\mathrm{R}\,y), r(y\,\mathrm{R}\,x)] \geq 0$ is true for all pair $(x, y) \in A^2$.

Property 3.1 (Weak completeness of $\geqslant$ on A)**.**
The global 'at least as good as' *relation,* $\geqslant$*, characterized by a bipolar credibility function* $r(x \geqslant y)$ *as defined in Definition 3.1, is* weakly complete *on* A*.*

Proof. From Formula (1) it is readily noticed (see Table 1) that, for all positions of the difference $x_i - y_i$ w.r.t. the preference discrimination thresholds, there appear five zones, labeled I to V, specifying the potential value of the credibility functions $r(x \geqslant_i y)$ and $r(y \geqslant_i x)$. Call W_I (resp. W_{II},W_{III},W_{IV} and W_V) the sum of

$x_i - y_i$		$-p_i$		$-q_i$	0	q_i		p_i	
position	I		II		III		IV		V
$r(x \geqslant_i y)$	-1		0		1		1		1
$r(y \geqslant_i x)$	1		1		1		0		-1

Table 1: Zones specifying the values of the marginal credibility functions $r(x \geqslant_i y)$ and $r(x \geqslant_i y)$ for all values of the difference $x_i - y_i$.

the weights of the criteria i such that $x_i - y_i$ belongs to zone I (resp. II, III, IV or V) as defined in Table 1. We have $r(x \geqslant y) = -W_I + W_{III} + W_{IV} + W_V$. Symmetrically, $r(y \geqslant x) = W_I + W_{II} + W_{III} - W_V$. Therefore, $r(x \geqslant y) + r(y \geqslant x)$ $= W_{II} + 2W_{III} + W_{IV} \geq 0$, and, hence, $\max(r(x \geqslant y), r(y \geqslant x))$ must be greater or equal 0. □

Only four exclusive preferential situations can thus appear when comparing two alternatives x and y on the family of criteria:

1. $r(x \geqslant y) > 0$ and $r(y \geqslant x) \leq 0$:
   x appears to be *globally at least as good as* y and not the converse;

2. $r(y \geqslant x) > 0$ and $r(x \geqslant y) \leq 0$:
   y appears to be *globally at least as good as* x and not the converse;

3. $r(x \geqslant y) > 0$ and $r(y \geqslant x) > 0$:
   both, x and y, appear to be *globally at least as good as one another*;

4. $r(x \geqslant y) = 0 = r(y \geqslant x)$:
   and, finally, the preferential situation between x and y appears to be *indeterminate*.

For each criterion $i \in F$, we can similarly state a marginal '*better than*' situation between any two alternatives x and y of A. We therefore reuse the same performance discrimination thresholds, q_i and p_i, to characterize a double threshold relation, $>_i$, whose bipolar-valued credibility, $r(>_i)$, is defined as follows:

$$r(x >_i y) \quad := \quad \begin{cases} \;\;1 & \text{if} \quad x_i - p_i \;\geq\; y_i \;, \\ -1 & \text{if} \quad x_i - q_i \;\leq\; y_i \;, \\ \;\;0 & \text{otherwise} \;. \end{cases} \tag{3}$$

Again, the credibility of the overall '*better than*' statement, $r(x > y)$, is computed by balancing the weights of the affirmative criteria against the weights of the refutative ones.

$$r(x > y) \quad := \quad \sum_{i \in F} \big[w_i \cdot r(x >_i y) \big]. \tag{4}$$

Following from the weak completeness of the global '*at least as good as*' relation (see Property 3.1), we notice that the incomparability part of the so-characterized global '*better than*' relation represents, in fact, the symmetric part of the global '*at least as good as*' relation. In other terms, the overall '*better than*' relation modeled by $r(x > y)$ on A is the codual, meaning the negation of the converse, of the overall '*at least as good as*' relation modeled by $r(x \geqslant y)$ on A.

Property 3.2 (Coduality of $>$ and $\geqslant$)**.**
The credibility of the asymmetric part $\gneqq$*, i.e.* $(x \geqslant y)$ *and* $(y \ngeqslant x)$*, of the overall* 'at least as good as' *relation* $\geqslant$ *on* A *is identical to the credibility of the overall* 'better than' *relation,* $>$ *on* A *and its codual* $(y \ngeqslant x)$*, i.e.* $r(x \gneqq y) = r(x > y) = -r(y \geqslant x)$*.*

Proof. Indeed, for each (x, y) in $A \times A$, from Formula 3 it follows readily that:

$$r(y \ngeqslant_i x) = \begin{cases} -1 & \text{if} \quad y_i + q_i \;\geq\; x_i \;, \\ \;\;1 & \text{if} \quad y_i + p_i \;\leq\; x_i \;, \\ \;\;0 & \text{otherwise} \end{cases}$$

Hence (see Table 2), Formulas (2) and (4) give the same credibility degrees to the codual $(\neg(\geqslant^{-1}) \;\equiv \nleqslant)$ and the asymmetric part $(\gneqq)$ of the overall '*at least as good as*' relation, as well as the corresponding overall '*better than*' relation $(>)$. □

$x_i - y_i$		$-p_i$	$-q_i$ 0 q_i		p_i
position	*I*	*II*	*III*	*IV*	*V*
$r(x \geqslant_i y)$	-1	0	1	1	1
$r(x \not\leqslant_i y)$	-1	-1	-1	0	1
$r(x \gneqq_i y)$	-1	-1	-1	0	1
$r(x >_i y)$	-1	-1	-1	0	1

Table 2: Zones specifying the potential values of marginal credibility functions for all values of the difference $x_i - y_i$.

Furthermore, apart from the global '*at least as good as*' relation, the classical outranking concept also takes into account '*veto*' situations [7, 2], i.e. situations where incommensurable large negative performance differences are observed when comparing the marginal performances of two decision alternatives.

3.2 Taking into account veto situations

In a pairwise comparison of performances, this feature is motivated by a concern to avoid short majority of small positive performance differences outweighing some very large negative performance differences. In order to identify the situation when such a marginal '*veto*' situation between two alternatives x and y of A is observed [2], we associate a *veto* (v_i) discrimination threshold with each criterion i's performance scale $[0; M_i]$ such that $p_i < v_i \leqslant M_i + \epsilon$.

We may thus define on each criterion $i \in F$ a threshold relation, denoted $\ll_i$, representing pairwise marginal '*considerably worse performing than*' statements on criterion i, and whose credibility, $r(\ll_i)$, is defined as follows:

$$r(x \ll_i y) \quad := \quad \begin{cases} \;\;1 & \text{if} \quad x_i + v_i \leqslant y_i \,, \\ -1 & \text{otherwise}\,. \end{cases}$$

It is worthwhile noticing here that, in the case of $v_i = M_i + \epsilon$, the criterion i does not support any veto principle.

Definition 3.2. The credibility of a global '*veto*' statement, denoted $r(x \ll y)$, may hence be defined by the overall disjunction of marginal '*considerably less performing than*' statements:

$$r(x \ll y) \quad := \quad r\Big(\bigvee_{i \in F} (x \ll_i y)\Big) \;=\; \max_{i \in F} \big[r(x \ll_i y)\big]. \tag{5}$$

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With Definitions 3.1 and 3.2, we are now ready to assess the credibility of a classical *outranking* statement.

3.3 The classical outranking statement

Following the genuine definition of the outranking concept [7, 9], we may state that:

Definition 3.3. An alternative x *outranks* an alternative y, denoted $x \succcurlyeq y$, when

1. a weighted *majority* of criteria validate the statement that x is performing at least as good as y, i.e. $x \geqslant y$;
2. and there is no veto raised against this validation, i.e. $x \not\ll y$.

Translating this classical outranking definition into the terms of our credibility calculus gives:

$$r(x \succcurlyeq y) \quad := \quad r\big(\,(x \geqslant y)\,\wedge\,(x \not\ll y)\,\big) \;=\; \min\big(\,r(x \geqslant y), r(x \not\ll y)\,\big) \tag{6}$$

We may now formally illustrate the potential hiatus between the classical outranking and its strict part.

Property 3.3 (Potential hiatus between $\succcurlyeq$ and $\succneqq$)**.**
Let $\succcurlyeq$ be the relation modeled by the classical outranking statement as defined above.

1. *The asymmetric part $\succneqq$ of the classical outranking relation $\succcurlyeq$, i.e. $x \succcurlyeq y$ and $y \not\succcurlyeq x$, is usually not identical to its corresponding codual relation $\not\preccurlyeq$* (Pirlot and Bouyssou [1])*;*
2. *Apart from the unanimous case, where $r(x \geqslant_i y) \pm 1$ for all criteria i, the absence of any veto situation $\ll_i$, for i in F, is a sufficient and necessary condition for making the credibility of $\succneqq$ identical to the credibility of $\not\preccurlyeq$.*

Proof.

(1) $r(x \succneqq y) = \min\big(r(x \succcurlyeq y), r(y \not\succcurlyeq x)\big) \;\leqslant\; r(y \not\succcurlyeq x) = r\big[\neg\big(\,(y \geqslant x) \wedge (y \not\ll x)\,\big)\big]$ $= r\big[\,(y \not\geqslant x) \,\vee\, \neg(y \not\ll x)\,\big] = r\big[\,(x > y) \,\vee\, (y \ll x)\,\big]$; the strict inequality appearing whenever $r(y \ll x) = 1$ and $r(x \succcurlyeq y) < 1$. In this case, $r(y \not\succcurlyeq x) = 1$, and hence, $r(x \succneqq y) = r(x \succcurlyeq y) < 1$.

(2) ($\Rightarrow$): $v_i = M_i + \epsilon$ for all $i \in F$ implies that $r(x \succcurlyeq y) = r(x \geqslant y)$ and the claimed identity follows from Property 3.2. ($\Leftarrow$): Conversely, observing $r(x \succneqq y)$ identical to $r(x \not\preccurlyeq y)$ implies for all $r(x \succcurlyeq y) < 1$ that $\min\big(r(x \succcurlyeq y), r(y \not\succcurlyeq x)\big) = r(y \not\succcurlyeq x)$. Hence $r(x \ll y)$ must necessarily admit the value -1 in all these cases.

□

As was recently reported by Pirlot and Bouyssou [1], this potential hiatus between the asymmetric part of the classical outranking relation and its codual raises a serious concern with respect to the logical soundness of Definition 3.3. Indeed, without this property, the correct strict preference statement cannot be deduced from the asymmetric part of a given outranking relation. From the point of view of the preferential semantics of the classical outranking concept, in the case of non unanimous concordance, only the complete absence of any veto mechanism can guarantee this commonly expected property of converse outranking statements. But, thus rejecting the whole veto principle is obliterating the very interest of Roy's original outranking concept itself [7], namely the conjunction of a weighted majority of 'at least as good as' statements (its *concordance* principle) and the absence of a veto situation (its *discordance* principle).

In the next section, to solve this conundrum, we will propose a new bipolar definition of the outranking concept which will allow us to overcome the hiatus problem observed with the classical outranking relation.

4 Bi-polarizing with large performance differences

From the proof of Property 3.3, we understand that, in order to overcome the previous hiatus and recover the full expressive power of the original outranking concept, it is in fact the unipolar veto principle that has to be put into a bipolar epistemic setting.

4.1 Taking into account large positive and negative performance differences

On each criterion i in F, let v_i ($p_i < v_i \leqslant M_i + \epsilon$) denote again a large performance difference that appears, in the eyes of the decision maker, non-compensable with opposed performance differences potentially observed on other criteria. We denote $x \lll_i y$ such a marginal '*considerably worse performing than*' statement on criterion

i, when the counter-performance of x with respect to y is larger or equal to v_i. Its bipolar-valued credibility function r is defined as follows:

$$r(x \lll_i y) \quad := \quad \begin{cases} \;\;1 & \text{if} \quad x_i + v_i \;\leqslant\; y_i \;, \\ -1 & \text{if} \quad x_i - v_i \;\geqslant\; y_i \;, \\ \;\;0 & \text{otherwise.} \end{cases} \tag{7}$$

The converse statement, denoted $\ggg_i$, concerns the corresponding '*considerably better performing than*' situation.

Following from the semantics of the (strict) negation in our bipolar credibility calculus, the above defined $\lll_i$ and the $\ggg_i$ statements define on A two binary relations that are the codual one of the other. Indeed, from Formula (7) (see Table 3)

$x_i - y_i$		$-v_i$ $\quad 0 \quad$ v_i	
position	I	II	III
$r(x \lll_i y)$	1	0	-1
$r(x \ggg_i y)$	-1	0	1

Table 3: Zones specifying the values of the marginal credibility functions $r(x \lll_i y)$ and $r(x \ggg_i y)$ for all values of the difference $x_i - y_i$.

it is readily noticed that $r\big(x \not\lll_i y\big) = -r(x \lll_i y) = r(x \ggg_i y)$. In case $v_i = M_i + \epsilon$, criterion i will show neither $\lll_i$, nor $\ggg_i$ situations.

When making apparent the hiatus between the asymmetric part and the codual of the classical outranking relation, we have noticed that the conjunctive handling of a veto situation is followed by a disjunctive handling of the corresponding converse counter-veto situation. Computing the bipolar-valued credibility of a global '*veto*' situation $\lll$ (respectively its dual 'counter-veto' situation $\ggg$) must therefore be modeled not as logical conjunction, but as an epistemic polarizing of all marginal $\lll_i$ statements:

$$r(x \lll y) \quad := \quad \circledvee_{i \in F}\big[r(x \lll_i y)\big]. \tag{8}$$

where $\circledvee$ represents the *bipolar sharpening* operator, also called *'symmetric disjunction'* [10, 11], and is defined as follows: $\circledvee_{i \in F}[r_i]$ equals $\max_{i \in F}(r_i)$ if $r_i \geqslant 0$ for all $i \in F$; $\min_{i \in F}(r_i)$ if $r_i \leqslant 0$ for all $i \in F$, and 0 otherwise. It is important to notice that $\circledvee$, like a common average operator, is not generally associative. To make the computation with $\circledvee$ unambiguous, we therefore always first compute sub-results for all positive and all negative terms, before assessing the final global result.

We may thus observe that $r(x \lll y) = 1$ if there exists a criterion i in F such that $r(x \lll_i y) = 1$, and there does not exist any criterion j in F such that

$r(x \ggg_j y) = 1$. Or, conversely, $r(x \ggg y) = 1$ if there exists a criterion i in F such that $r(x \ggg_i y) = 1$, and there does not exist any criterion j in F such that $r(x \lll_j y) = 1$.

It is worthwhile noticing that this bipolar formulation of the veto principle indeed verifies the requested coduality property.

Property 4.1 (Coduality of the bipolar veto credibility)**.**
The credibility of the codual, $r(\not\ggg)$, respectively $r(\not\lll)$, is identical to the credibility of the global 'considerably better performing than' *statement, $r(\ggg)$, respectively the global* 'considerably worse performing than' *statement $r(\lll)$.*

Proof. It is sufficient to recall that the marginal statements $\lll_i$ and $\ggg_i$ are codual one to the other (see Table 3) and to notice that the $\circledcirc$ operator used in Formula (8) is self-dual. □

4.2 The bipolar outranking statement

With this bipolar modeling of potential veto and counter-veto situations, we are now able to restate outranking situations in such a way that the hiatus between their asymmetric part and their codual do no longer appear.

Definition 4.1 (Bipolar outranking)**.**
When x and y are two given decision alternatives, we say that:

1. '*x outranks y*', denoted $x \succsim y$, if a weighted majority of criteria validate a global outranking situation between x and y and no considerable counter-performance is observed on a *discordant* criterion,

2. '*x does not outrank y*', denoted $x \not\succsim y$, if only a weighted minority of criteria validate a global outranking situation between x and y and no '*considerably better*' performance is observed on a *concordant* criterion,

3. the statement '*x outranks y*' can *neither be validated nor invalidated* if we conjointly observe some '*considerably worse*' as well as some '*considerably better*' performances.

In terms of our bipolar credibility calculus the credibility $r(x \succsim y)$ of such a bipolar outranking statement, $x \succsim y$, may be computed as follows[2]:

$$r(x \succsim y) \quad := \quad \big[\, r(x \geqslant y) \circledcirc_{i \in F} r(x \not\lll_i y) \,\big]\,. \tag{9}$$

[2]The $\circledcirc$ operator not being associative, we may not directly define the credibility $r(x \succsim y)$ to be $r(x \geqslant y) \circledcirc r(x \not\lll y)$. Instead, the notation in Formula (9) is a shortcut for $\big[\, r(x \geqslant y) \circledcirc r(x \not\lll_1 y) \circledcirc \ldots \circledcirc r(x \not\lll_n y) \,\big]$.

If, on the one hand, $v_i = M_i + \epsilon$ for all $i \in F$, i.e. in the absence of any vetoes, we recover the classical outranking case, where $r(x \succsim y) = r(x \succcurlyeq y) = r(x \geqslant y)$. If, on the other hand, we observe conjointly considerably better and worse performances on some criteria i and j, the validation or invalidation of the outranking statement '$x \succsim y$' gets doubtful. Consequently, its bipolar credibility is put to 0, i.e. the indeterminate case. Here the classical outranking would have been certainly invalidated instead.

If we observe, however, solely some considerably better performing situations ($r(x \ggg_i y) = 1$ for some $i \in F$), coupled with a weighted majority of validating criteria ($r(x \geqslant y) \geq 0$), the outranking statement appears definitely validated: $r(x \succsim y) = 1$.

We may, furthermore, solely observe some considerably worse performing situations ($r(x \lll_i y) = 1$ for some $i \in F$), coupled with a minority only of validating criteria ($r(x \geqslant y) \leq 0$), the outranking situation appears definitely invalidated: $r(x \succsim y) = -1$.

In all other remaining cases, i.e. – a minority only of validating criteria, but, some considerably better performances observed otherwise, or, – a majority of validating criteria, but, some considerably worse performances observed otherwise, the outranking statement, due to the non-compensable large positive and negative performances differences we observe, may neither be validated, nor, invalidated anymore: consequently $r(x \succsim y) = 0$.

4.3 Properties of the bipolar outranking relation

Apart from being trivially reflexive [12]. i.e. $r(x \succsim x) = 1$ for all $x \in A$, let us first notice that the bipolar outranking relation characterized by Formula (9), contrary to the classical outranking, does not loose the weak completeness property of the underlying global '*at least as good as*' relation when getting bi-polarized with large performance differences.

Property 4.2 (weak completeness of $\succsim$ on A).
The bipolar outranking relation, $\succsim$, characterized by its credibility function $r(\succsim)$, computed following Formula (9), is weakly complete.

Proof. The property follows directly from the weak completeness of the underlying global '*at least as good as*' relation (see Property 3.1) and the observation that the polarizing effect is limited to potentially transforming some positive or negative credibility degrees into indeterminate ones. □

Consequently, the credibility of the codual $\not\precsim$ of the bipolar outranking relation $\succsim$ will indeed be equal to the credibility of its asymmetric part $\succnsim$.

Property 4.3 (Codual equals asymmetric part)**.**
If $\succnsim$ denotes the asymmetric part of the bipolar outranking $\succsim$, and $\not\precsim$ its corresponding codual relation, then:

$$r(x \succnsim y) \;=\; \min\big(\, r(x \succsim y), r(y \not\succsim x)\,\big) \;=\; r(x \not\precsim y)\,, \quad \forall (x,y) \in A^2. \tag{10}$$

Proof. When comparing any pair (x,y) of alternatives, three exclusive cases may be observed:

1. No marginal large performance differences do occur.
   By Property 3.2, $r\big(x \not\precsim y\big) \;=\; r(x \not\leqslant y) \;=\; r(x > y) \;\leq\; r(x \geqslant y)$.
   Hence $r(x \succnsim y) = \min\big(\, r(x \geqslant y), r(x > y)\,\big) = r(x \not\precsim y)$.

2. Large positive and negative performance differences are conjointly observed.
   Hence, $r(x \not\precsim y) \;=\; 0 \;=\; r(x \succnsim y)$
   (0 is the negational fixpoint of the bipolar-valued credibility calculus).

3. Only large positive (resp. large negative) performance differences are observed.
   Again,

$$\begin{aligned}
r(x \not\precsim y) \;&=\; -\big[\, r(x \leqslant y) \;\varovee_{i\in F}\; r(x \not\ggg_i y)\,\big], && \text{(the negation of Formula (9))}\\
&=\; \big[\, r(x \not\leqslant y) \;\varovee_{i\in F}\; -r(x \lll_i y)\,\big], && \text{(by Property 4.1)}\\
&=\; \big[\, r(x > y) \;\varovee_{i\in F}\; r(x \not\lll_i y)\,\big]. && \text{(by Properties 3.2 and 4.1)}
\end{aligned}$$

$\square$

This last property greatly enhances the efficiency and usefulness of outranking approaches for solving selection, ranking or sorting decision aid problems. Sound semantics of the negation of the bipolar outranking statement may give now direct access to the corresponding converse strict preference statement.

In a last section, we will illustrate the usefulness of this feature on a small multiple criteria best choice recommendation problem.

criteria	w_i	a_1	a_2	a_3	a_4	a_5
g_1	4/13	17.3	25.5	**76.1**	32.7	0.8
g_2	1/13	7.3	5.7	53.0	**99.7**	91.3
g_3	4/13	90.3	79.8	**93.0**	48.2	11.4
g_4	2/13	13.2	**94.1**	48.9	49.9	19.6
g_5	2/13	42.4	16.1	64.8	78.2	**81.1**

Table 4: Randomly generated performance tableau

5 Numerical illustration

Let us consider in Table 4 the randomly generated performance evaluations of five potential decision alternatives: a_1, a_2, ..., a_5, on a supposedly coherent family of five criteria: g_1, g_2, ... , g_5. All criteria: g_1, g_2, ..., g_5 , with given significance weights w_i (see Table 4, column 2), use a real performance scale running from 0.0 (worst level) to 100.0 (best level) and admitting three discrimination thresholds: *indifference* ($\pm$10.0), *preference* (+20.0) and *considerably better or worse performing* ($\pm$60.0).

5.1 Global '*at least as good as*' credibility

In Table 5 we show the bipolar-valued credibility of the overall '*at least as good as*' statements, $r(\geqslant)$ (see Definition 3.1) as integer multiples of 1/13. Below each credibility, which may thus take any integer value between -13 and $+13$, we mention in brackets the number of large positive ($+n$) and/or negative ($-n$) performance differences (≥ 60.0) we observe in the pairwise comparisons of the marginal performances on each criterion.

Let us first notice that – due to the potential asymmetric and indeterminate part of the $\geqslant$ relation – the symmetric $r(\geqslant)$ credibility values are usually neither equal nor of opposite sign: $r(a_1 \geqslant a_2) = +9/13$ for instance, whereas $r(a_2 \geqslant a_1) = +5/13$ (see Table 7). However, $\lll$ and $\ggg$ being codual to each other (see Property 4.1), the symmetric numbers of large performance differences are always of opposite sign: $r(a_1 \lll a_2) = -1$ implies $r(a_2 \ggg a_1) = +1$ for example. We may notice, furthermore, that the pairwise comparison between a_1 and a_5 (see Table 5 row a_1 right end position) shows conjointly a positive and a negative large performance difference. The comparison between a_2 and a_5 reveals even two positive and two negative large performance differences.

$\mathbf{r(x \geqslant y) \cdot 13}$	a_1	a_2	a_3	a_4	a_5
a_1	–	$+\mathbf{9}$ $(-\mathbf{1})$	-5	-5 (-1)	$+7$ $(-\mathbf{1}, +\mathbf{1})$
a_2	$+\mathbf{5}$ $(+\mathbf{1})$	–	-5	$+7$ (-2)	$+7$ $(+\mathbf{2}, -\mathbf{2})$
a_3	$+13$	$+9$	–	$+9$	$+9$ $(+2)$
a_4	$+5$ $(+1)$	$+1$ $(+2)$	-3	–	$+13$
a_5	$+1$ $(+1, -1)$	-7 $(+2, -2)$	-7 (-2)	-7	–

Table 5: Pairwise global *'at least as good as'* credibility degrees with, in brackets below, the number of large positive and/or negative performance differences

5.2 Polarizing with large performance differences

In Table 6, we show the eventual polarization of the $\geqslant$ credibility degrees that results from taking into account all negative, as well as positive, large performance differences. We indicate in brackets the corresponding classical veto polarization $r(x \succcurlyeq y)$

$\mathbf{r(x \succsim y)}$	a_1	a_2	a_3	a_4	a_5
a_1	–	$\mathbf{0.0}$ $(-\mathbf{1.0})$	$-.38$	-1.0	0.0 (-1.0)
a_2	$+\mathbf{1.0}$ $(+.38)$	–	$-.38$	0.0	0.0 (-1.0)
a_3	$+\mathbf{1.0}$	$+\mathbf{.69}$	–	$+\mathbf{.69}$	$+\mathbf{1.0}$ $(+.69)$
a_4	$+\mathbf{1.0}$ $(+.38)$	$+\mathbf{1.0}$ $(+.08)$	$-.23$	–	$+\mathbf{1.0}$
a_5	0.0 (-1.0)	0.0 (-1.0)	-1.0	$-.54$	–

Table 6: Bipolar outranking credibility degrees, $r(x \succsim y)$, with corresponding classical outranking credibility degrees in brackets when different

when it is different from the actual bipolar $r(x \succsim y)$ one.

Let us now reconsider in detail the comparison between a_1 and a_2 as shown in Table 7. A highly significant majority of nearly 85% [3], indeed all criteria except g_4, validate the '*at least as good as*' statement (see column 5). However, on criterion g_4, a_1 shows a considerable counter-performance of -80.9 when compared to a_2 – a

[3]The conversion from the bipolar-valued credibility degree to a simple majority percentage is computed as follows: $(r(\geqslant)+1.0)/2.0$, i.e. a positive shift of one and a rescaling to the unit interval. For instance $r(a_1 \geqslant a_2) = 9/13$ corresponds to a majority of $(9 + 13)/26 = 11/13 \approx 84.6\%$.

criteria	w_i	a_1	a_2	$\Delta(a_1, a_2)$	$r(a_1 \geqslant_i a_2)$	$\Delta(a_2, a_1)$	$r(a_2 \geqslant_i a_1)$
g_1	4/13	17.3	25.5	-8.2	$+4/13$	$+8.2$	$+4/13$
g_2	1/13	7.3	5.7	$+1.6$	$+1/13$	-1.6	$+1/13$
g_3	4/13	90.3	79.8	$+10.5$	$+4/13$	-10.5	$0/13$
g_4	2/13	13.2	94.1	$-\mathbf{80.9}$	$-2/13$	$+\mathbf{80.9}$	$+2/13$
g_5	2/13	42.4	16.1	$+26.3$	$+2/13$	-26.3	$-2/13$
global	13/13	-	-	$r(a_1 \geqslant a_2)$	$= +\mathbf{9/13}$	$r(a_2 \geqslant a_1)$	$= +\mathbf{5/13}$

Table 7: Comparison of marginal performances of alternatives a_1 and a_2 with constant indifference (10.0), preference (20.0) and veto (60.0) discrimination thresholds.

performance difference much larger than the supposed large non-compensable performance difference threshold of 60.0. Following the classical veto principle, we would therefore definitely invalidate this outranking. But, as we have seen in Property 3.3, this definite invalidation in fact breaks the identity between the asymmetric part and the codual of the global outranking relation. In the bipolar setting, this contradictory observation, namely a weighted majority validating coupled with a considerable counter-performance, will instead raise doubts against the validation, *as well as the invalidation*, of this outranking statement. Consequently, $r(a_1 \succsim a_2)$ is put to 0.0, the indeterminate value.

An even more indeterminate situation can be observed when comparing the performances of alternative a_1 with those of alternative a_5 (see Table 4). Here, all criteria, except g_2, with a significance of 77%, validate the outranking situation. Furthermore, a considerably better performance of +79.8 on criterion g_3 further reinforces this validation. However, on g_2, we also notice a counter-performance that is again much larger (-84.1) than the non-compensable difference threshold of 60.0. Again, the classical outranking approach would definitely invalidate this outranking, whereas in the bipolar approach, we prefer to doubt any validation – as well as invalidation – of such outranking statements. In an incommensurable approach, we definitely cannot know what the compensation of a very large positive with a very large negative performance may give as preferential result. It is therefore not appropriate to validate this statement, as it is not to invalidate it. In the absence of compensable performance measures on each criterion, and to be prudent in our preferential constructions, we may effectively only retain the determined preferential judgments.

Finally, let us consider those pairwise comparisons where a weighted majority of criteria indeed validate an outranking, and where we also observe some considerably better performances. In Table 5, we notice, for instance, the compelling comparisons between a_2 and a_1, between a_3 and a_5, and between a_4 and a_1 or a_2. In all these cases, we can comprehensively validate the corresponding outranking statements, a fact not taken into account in the implementation of the classical outranking concept.

5.3 Recommending a best choice

In order to solve the given best choice recommendation problem, let us compute the codual of the bipolar outranking shown in Table 6. Through Property 4.3, this will give us directly the asymmetric strict part of the bipolar outranking, meaning the credibility degrees of the corresponding '*better than*' statements on A, polarized the case given with large performance differences as shown in Table 8.

$\mathbf{r(x \gtrsim\!\!\!\!/\; y) \cdot 13}$	a_1	a_2	a_3	a_4	a_5
a_1	–	-13	-13	-13	0
a_2	0	–	-9	-13	0
$\mathbf{a_3}$	$\mathbf{+5}$	$\mathbf{+9}$	–	$\mathbf{+3}$	$\mathbf{+13}$
a_4	$+13$	0	-9	–	$+7$
a_5	0	0	-13	-13	–

Table 8: Credibility degrees of the asymmetric (strict) part of bipolar outranking.

In the first row, we observe that alternative a_1 performs less well than alternatives a_2, a_3 and a_4, but that the outranking situation with a_5 is indeterminate. Similarly in the second and the last row, we observe that alternatives a_2, respectively a_5, perform less well than a_3 and a_4 and the situation with a_1, respectively a_5, is again indeterminate. Alternative a_4 performs also less well than a_3. Only a_3 appears to perform *positively better than all* the other potential decision alternatives. Alternative a_3 consequently gives what is commonly called a CONDORCET *winner*: namely a_3 outperforms all the other alternatives with a weighted majority of significance of at least $(3/13 + 1)/2 \approx 61.5\%$. In Table 4, we can indeed verify that, on both the most important criteria (g_1 and g_3) and cumulating a majority of significance of $8/13 \approx 61.5\%$, alternative a_3 shows without doubt by far the best performances. And as such, it evidently represents here the *best choice recommendation*.

6 Conclusion

In this paper, we have introduced a new bipolar veto and counter-veto principle which allows us to construct a bipolarly extended global outranking relation that guarantees the formal identity between its strict (asymmetric) part and the negation of its converse relation.

We have shown that the classical outranking definition in fact models the difficulty to compensate excellent performances with considerable counter-performances as incomparability situations. It appears that this approach introduces however, a hiatus between the asymmetric part and the codual of the outranking relation. To overcome this logical unsoundness, we prefer to rely here on the indeterminacy value of our bipolar-valued credibility calculus for expressing our doubts concerning the meaningfulness of a numerical compensation of such contrasted performances. Using the bipolar outranking concept, we may recover both the weak completeness and the coduality property of the underlying global '*at least as good as*' relation.

As a consequence of this new conceptual design, we preserve, on the one hand, all the original and interesting concordance versus discordance semantics of the classical outranking concept, whilst, on the other hand, we do not loose the sound semantics of the logical negation of the actual pairwise outranking statement. Thus appears a more prudent and robust outranking relation, allowing the decision aid practice to coherently take into account large non-compensable performance differences which may considerably question otherwise significant global '*at least as good as*' statements.

Enhanced and simpler operational semantics for selecting the best choice [8], or quick multiple criteria sorting, then become potentially available. Also, inverse multiple criteria decision analysis, where preferential model parameters like criteria weights are inferred from indirect observation of global preferences [13], may now effectively tackle large non-compensable performances differences effectively and thereby use all the expressive power of the genuine outranking approach.

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Index of relational notation

$\geqslant_i$	Marginal '*at least as good as*' relation on criterion i
$\geqslant$	Global '*at least as good as*' relation
$>_i$	Marginal '*better than*' relation on criterion i
$>$	Global '*better than*' relation
$\not\leqslant$	Codual (negation of the converse) of the $\geqslant$ relation
$\ll_i$	Marginal classical '*considerably worse performing than*' relation
$\ll$	Classical '*veto*' relation
$\succcurlyeq$	Classical '*outranking*' relation
$\succneqq$	Asymmetric part of the classical '*outranking*' relation
$\not\preccurlyeq$	Codual (negation of the converse) of the $\succcurlyeq$ relation
$\lll_i$	Marginal bipolar '*considerably worse performing than*' relation
$\ggg_i$	Marginal bipolar '*considerably better performing than*' relation
$\lll$	Global bipolar '*considerably worse performing than*' (veto) relation
$\ggg$	Global bipolar '*considerably better performing than*' (counter-veto) relation
$\succsim$	Bipolar '*outranking*' relation
$\succnsim$	Asymmetric part of the bipolar '*outranking*' relation
$\not\precsim$	Codual (negation of the converse) of the $\succsim$ relation

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