Preference aggregation with multiple criteria of ordinal significance

Raymond Bisdorff*

Abstract

In this paper we address the problem of aggregating outranking situations in the presence of multiple preference criteria of ordinal significance. The concept of ordinal concordance of the global outranking relation is defined and an operational test for its presence is developed. Finally, we propose a new kind of robustness analysis for global outranking relations taking into account classical dominance, ordinal and classical majority concordance in a same ordinal valued logical framework.

Key words : Multicriteria aid for decision, ordinal significance weights, robust outranking

1 Introduction

Commonly the problem of aggregating preference situations along multiple points of view is solved with the help of cardinal weights translating the significance the decision maker gives each criteria (Roy and Bouyssou, 1993). However, determining the exact numerical values of these cardinal weights remains one of the most obvious practical difficulty in applying multiple criteria aid for decision (Roy and Mousseau, 1996).

To address precisely this problem, we generalize in a first section the classical concordance principle, as implemented in the Electre methods (Roy, 1985), to the context where merely ordinal information concerning these significance of criteria is

*Operations Research Unit, University of Luxembourg, Campus Limpertsberg, Avenue de la Faïencerie, L-1511, Luxembourg. raymond.bisdorff@uni.lu

available. Basic data and notation is introduced and the classical cardinal concordance principle is reviewed. The ordinal concordance principle is formally introduced and illustrated on a simple car selection problem.

In a second section, we address theoretical foundations and justification of the definition of ordinal concordance. By the way, an operational test for assessing the presence or not of the ordinal concordance situation is developed. The core approach involves the construction of a distributional dominance test similar in its design to the stochastic dominance approach.

In a last section we finally address the robustness problem of multricriteria decision aid recommendations in the context of the choice problematics. Classical dominance, i.e. unanimous concordance, ordinal as well as cardinal majority concordance are considered altogether in a common logical framework in order to achieve robust optimal choice recommendation. We rely in this approach on previous work on good choices from ordinal valued outranking relations (see Bisdorff and Roubens, 2003).

2 The ordinal concordance principle

We start with setting up the necessary notation and definitions. We follow more or less the notation used in the French multicriteria decision aid community.

2.1 Basic data and notation

As starting point, we require a set A of potential decision actions. To assess binary outranking situations between these actions we consider a coherent family $F = \{g_1, \ldots, g_n\}$ of n preference criteria (Roy and Bouyssou, 1993, Chapter 2).

The performance tableau gives us for each couple of decisions actions $a, b \in A$ their corresponding performance vectors $g(a) = \big(g_1(a), ..., g_n(a)\big)$ and $g(b) = \big(g_1(b), ..., g_n(b)\big)$.

A first illustration, shown in Table 1, concerns a simple car selection problem taken from Vincke (1992, pp. 61–62)). We consider here a set $A = \{m_1, \ldots, m_7\}$ of potential car models which are evaluated on four criteria: *Price*, *Comfort*, *Speed* and *Design*. In this supposedly coherent family of criteria, the *Price* criterion works in the negative direction of the numerical amounts. The evaluations on the qualitative criteria such as *Comfort*, *Speed* and *Design* are numerically coded as follows: 3 means *excellent* or *superior*, 2 means *average* or *ordinary*, 1 means *weak*.

In general, we may observe on each criterion $g_j \in F$ an indifference threshold $q_j \geq 0$ and a strict preference threshold $p_j \geq q_j$ (see Roy and Bouyssou, 1993,

Table 1: Car selection problem: performance tableau

Cars	q_j	p_j	m_1	m_2	m_3	m_4	m_5	m_6	m_7	w
1: Price	10	50	-300	-270	-250	-210	-200	-180	-150	5/15
2: Comfort	0	1	3	3	2	2	2	1	1	4/15
3: Speed	0	1	3	2	3	3	2	3	2	3/15
4: Design	0	1	3	3	3	2	3	2	2	3/15

Source:Vincke, Ph. 1992, pp. 61–62

pp. 55–59). We suppose for instance that the decision-maker admits on the *Price* criterion an indifference threshold of 10 and a preference threshold of 50 units.

To simplify the exposition, we consider in the sequel that all criteria support the decision maker's preferences along a positive direction. Let $\Delta_j(a,b) = g_j(a) - g_j(b)$ denote the difference between the performances of the decision actions a and b on criterion g_j. For each criterion $g_j \in F$, we denote "$a \, \mathrm{S}_j \, b$" the semiotic restriction of assertion "a outranks b" to the individual criterion g_j.

Definition 1. $\forall a, b, \in A$, the level of credibility $r(a \, \mathrm{S}_j \, b)$ of assertion "$a \, \mathrm{S}_j \, b$" is defined as:

$$r(a \, \mathrm{S}_j \, b) = \begin{cases} 1 & \text{if} \quad \Delta_j(a,b) \geq -q_j \\ \frac{p_j + \Delta_j(a,b)}{p_j - q_j} & \text{if} \quad -p_j \leq \Delta_j(a,b) \leq -q_j \\ 0 & \text{if} \quad \Delta_j(a,b) < -p_j. \end{cases} \quad (1)$$

The level of credibility $r(\overline{a \, \mathrm{S}_j \, b})$ associated with the truthfulness of the negation of the assertion "$a \, \mathrm{S}_j \, b$" is defined as follows:

$$r(\overline{a \, \mathrm{S}_j \, b}) = 1 - r(a \, \mathrm{S}_j \, b). \quad (2)$$

Following these definitions, we find in Table 1 that model m_6 clearly outranks model m_2 on the *Price* criterion ($\Delta_1(m_6, m_2) = 90$ and $r(m_6 \, \mathrm{S}_1 \, m_2) = 1$) as well as on the *Speed* criterion ($\Delta_3(m_6, m_2) = 1$ and $r(m_6 \, \mathrm{S}_3 \, m_2) = 1$).

Inversely, model m_2 clearly outranks model m_6 on the *Comfort* criterion as well as on the *Design* criterion. Indeed $\Delta_2(m_2, m_6) = 2$ and $r(m_2 \, \mathrm{S}_2 \, m_6) = 1$ as well as $\Delta_4(m_2, m_6) = 1$ and $r(m_2 \, \mathrm{S}_4 \, m_6) = 1$.

A given performance tableau, if constructed as required by the corresponding decision aid methodology (see Roy, 1985), is warrant for the truthfulness of these "*local*", i.e. the individual criterion based preferences of the decision maker. To assess however global preference statements integrating all available criteria, we need to aggregate these local warrants by considering the relative significance the decision-maker attributes to each individual criterion with respect to his global preference system.

2.2 The classical concordance principle

In the Electre based methods, this issue is addressed by evaluating if, yes or no, a more or less significant majority of criteria effectively concord on supporting a given global outranking assertion (see Roy and Bouyssou, 1993; Bisdorff, 2002). This classical majority concordance principle for assessing aggregated preferences from multiple criteria was originally introduced by Roy (1968).

Definition 2. Let $w = (w_1, ..., w_n)$ be a set of significance weights corresponding to the n criteria such that: $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. For $a, b \in A$, let $a \, \mathrm{S} \, b$ denote the assertion that "*a globally outranks b*"[1]. We denote $r_w(a \, \mathrm{S} \, b)$ the credibility of assertion $a \, \mathrm{S} \, b$ considering given significance weights w.

$$r_w(a \, \mathrm{S} \, b) \; = \; \sum_{j=1}^{n} \big(w_j \cdot r(a \, \mathrm{S}_j \, b) \big). \tag{3}$$

Assertion "$a \, \mathrm{S} \, b$" is considered *rather true than false*, as soon as the weighted sum of criterial significance in favour of the global outranking situation obtains a strict majority, i.e. the weighted sum of criterial significance is greater than 50%. To clearly show the truth-functional denotation implied by our credibility function r_w, we shall introduce some further notations.

Definition 3. Let "$a \, \mathrm{S} \, b$" denote the fact that *a globally outranks b*. We denote $\|a \, \mathrm{S} \, b\|_w$ the logical denotation of the credibility calculus taking its truth values in a three valued truth domain $L_3 = \{f_w, u, t_w\}$ where f_w means *rather false than true* considering importance weights w, t_w means *rather true than false* considering importance weights w and u means *logically undetermined.*

$$\|a \, \mathrm{S} \, b\|_w \; = \; \begin{cases} t_w & \text{if} \quad r_w(a \, \mathrm{S} \, b) > 0.5 \; ; \\ f_w & \text{if} \quad r_w(a \, \mathrm{S} \, b) < 0.5 \; ; \\ u & \text{otherwise.} \end{cases} \tag{4}$$

[1]Readers familiar with the outranking concept will notice the absence of the *veto* issue in our definition of the outranking situation. The veto principle, also called discordance principle by Roy, requires some measurable distance on the criteria scales. For robustness purposes we prefer to keep with solely the sound ordinal properties of the criterion function concept. And the concordance principle already naturally integrates a balancing reasons principle by weighting concordant against discordant arguments (see Bisdorff and Roubens, 2003)

In our example, let us suppose that the decision-maker admits the significance weights w shown in Table 1. The *Price* criterion is the most significant with a weight of 5/15. Then comes the *Comfort* criterion with 4/15 and finally, both the *Speed* and the *Design* criteria have identical weights 3/15. By assuming that the underlying family of criteria is indeed coherent, we may thus state that the assertion "$m_6 \mathrm{S}_w m_2$" with aggregated significance of 53.3% is *rather true than false* with respect to the given importance weights w.

The majority concordance approach obviously requires a precise numerical knowledge of the significance of the criteria, a situation which appears to be difficult to achieve in practical applications of multicriteria decision aid.

Substantial efforts have been concentrated on developing analysis and methods for assessing these cardinal significance weights (see Roy and Mousseau, 1992, 1996). Following this discussion, Dias and Clímaco (2002) propose to cope with imprecise significance weights by delimiting sets of potential significance weights and enrich the proposed decision recommendations with a tolerance in order to achieve robust recommendations.

In this paper we shall not contribute directly to this issue but rely on the fact that in practical application the ordinal weighting of the significance of the criteria are generally easier to assess and more robust than any precise numerical weights.

2.3 Ordinal concordance principle

Let us assume that instead of a given cardinal weight vector w we observe a complete pre-order π on the family of criteria F which represents the significance rank each criterion takes in the evaluation of the concordance of the global outranking relation S to be constructed on A.

In our previous car selection example, we may notice for instance that the proposed significance weights model the following ranking π: *Price* $>$ *Comfort* $>$ { *Speed, Design*}.

A precise set w of numerical weights may now be compatible or not with such a given significance ranking of the criteria.

Definition 4. w is a π-compatible set of weights if and only if:

> $w_i = w_j$ for all couples (g_i, g_j) of criteria which are of the same significance with respect to π;
>
> $w_i > w_j$ for all couples (g_i, g_j) of criteria such that criterion g_i is certainly more significant than criterion g_j in the sense of π.

We denote $W(\pi)$ the set of all π-compatible weight vectors w.

Definition 5. For $a, b \in A$, let "$a\,\mathrm{S}_\pi\,b$" denote the fact that *"a globally outranks b with a significant majority for every π-compatible weight vector"*.

$$a\ \mathrm{S}_\pi\ b \quad \Leftarrow \quad \big(\, r_w(a\,\mathrm{S}\,b) > 0.5,\ \forall w \in W(\pi)\,\big). \tag{5}$$

For short, we say that *a globally outranks b in the sense of the ordinal concordance principle.*

2.4 Theoretical justification

In other words, the $a\,\mathrm{S}_\pi\,b$ situation is given if for all π-compatible weight vectors w, the aggregated significance of the assertion $a\,\mathrm{S}_w\,b$ outranks the aggregated significance of the negation $\overline{a\,\mathrm{S}_w\,b}$ of the same assertion.

Proposition 1.

$$a\,\mathrm{S}_\pi\,b \quad \Leftarrow \quad \big(\, r_w(a\,\mathrm{S}\,b)\ >\ r_w(\overline{a\,\mathrm{S}\,b});\ \forall w \in W(\pi)\,\big). \tag{6}$$

Proof. Implication 6 results immediately from the observation that:

$$\sum_{g_j \in F} w_j \cdot r(a\,\mathrm{S}_j\,b)\ >\ \sum_{g_j \in F} w_j \cdot r(\overline{a\,\mathrm{S}_j\,b}) \quad \Leftrightarrow \quad \sum_{g_j \in F} w_j \cdot r(a\,\mathrm{S}_j\,b)\ >\ \frac{1}{2}.$$

Indeed, $\forall g_j \in F$ we observe that $r(a\,\mathrm{S}_j\,b) + r(\overline{a\,\mathrm{S}_j\,b}) = 1$. This fact implies that:

$$\sum_{g_j \in F} w_j \cdot r(a\,\mathrm{S}_j\,b)\ +\ \sum_{g_j \in F} w_j \cdot r(\overline{a\,\mathrm{S}_j\,b})\ =\ 1.$$

□

Coming back to our previous car selection problem, we shall later on verify that model m_6 effectively outranks all other 6 car models following the ordinal concordance principle, With any π-compatible set of cardinal weights, model m_6 will always outrank all other car models with a '*significant*' majority of criteria.

We still need now a constructive approach for computing such ordinal concordance results.

3 Testing for ordinal concordance

In this section, we elaborate general conditions that must be fulfilled in order to be sure that there exists an ordinal concordance in favour of the global outranking situation. By the way we formulate an operational procedure for constructing a relation S_π on A from a given performance tableau.

3.1 Positive and negative significance

The following condition is identical to the condition of the ordinal concordance principle (see Definition 5).

Proposition 2. *$\forall a, b \in A$ and $\forall w \in W(\pi)$:*

$$r_w(a\,\mathrm{S}\,b) > r_w(\overline{a\,\mathrm{S}\,b}) \;\Leftrightarrow\; r_w(a\,\mathrm{S}\,b) \,-\, r_w(\overline{a\,\mathrm{S}\,b}) \;>\; r_w(\overline{a\,\mathrm{S}\,b}) \,-\, r_w(a\,\mathrm{S}\,b). \qquad (7)$$

Proof. The equivalence between the right hand side of Equivalence 7 and the right hand side of Implication 6 is obtained with simple algebraic manipulations. □

The inequality in the right hand side of Equivalence 7 gives us the operational key for implementing a test for ordinal concordance of an outranking situation. The same weights w_j and $-w_j$, denoting the "*confirming*", respectively the "*negating*", significance of each criterion, appear on each side of the inequality.

Furthermore, the sum of the coefficients $r(a\,\mathrm{S}_j\,b)$ and $r(\overline{a\,\mathrm{S}_j\,b})$ on each side of the inequality is a constant equal to n, i.e. the number of criteria in F. Therefore these coefficients may appear as some kind of credibility distribution on the set of positive and negative significance weights.

3.2 Significance distributions

Suppose that the given pre-order π of significance of the criteria contains k equivalence classes which we are going to denote $\pi_{(k+1)}, ..., \pi_{(2k)}$ in increasing sequence. The same equivalence classes, but in in reversed order, appearing on the "negating" significance side, are denoted $\pi_{(1)}, ..., \pi_{(k)}$.

Definition 6. For each equivalence class $\pi_{(i)}$, we denote $w_{(i)}$ the cumulated negating, respectively confirming, significance of all equi-significant criteria gathered in this equivalence class:

$$i = 1, ..., k : w_{(i)} = \sum_{g_j \in \pi_{(i)}} -w_j; \quad i = k+1, ..., 2k : w_{(i)} = \sum_{g_j \in \pi_{(i)}} w_j. \qquad (8)$$

We denote $c_{(i)}$ for $i = 1, ..., k$ the sum of all coefficients $r(\overline{a\,\mathrm{S}_j\,b})$ such that $g_j \in \pi_{(i)}$ and $c_{(i)}$ for $i = k+1, ..., 2k$ the sum of all coefficients $r(a\,\mathrm{S}_j\,b)$ such that $g_j \in \pi_{(i)}$. Similarly, we denote $\overline{c_{(i)}}$ for $i = 1, ..., k$ the sum of all coefficients $r(a\,\mathrm{S}_j\,b)$ such that $g_j \in \pi_{(i)}$ and $\overline{c_{(i)}}$ for $i = k+1, ..., 2k$ the sum of all coefficients $r(\overline{a\,\mathrm{S}_j\,b})$ such that $g_j \in \pi_{(i)}$.

With the help of this notation, we may rewrite Equivalence 7 as follows:

Proposition 3. *$\forall a, b \in A$ and $w \in W(\pi)$:*

$$r_w(a\,\mathrm{S}\,b) > r_w(\overline{a\,\mathrm{S}\,b}) \quad \Leftrightarrow \quad \sum_{i=1}^{2k} c_{(i)} \cdot w_{(i)} > \sum_{i=1}^{2k} \overline{c_{(i)}} \cdot w_{(i)}. \tag{9}$$

Coefficients $c_{(i)}$ and $\overline{c_{(i)}}$ represent two distributions, one the negation of the other, on an ordinal scale determined by the increasing significance $w_{(i)}$ of the equivalence classes in $\pi_{(i)}$.

3.3 Ordinal distributional dominance

We may thus test the right hand side inequality of Equivalence 7 with the classical stochastic dominance principle originally introduced in the context of efficient portfolio selection (see Hadar and Russel, 1969; Hanoch and Levy, 1969).

We denote $C_{(i)}$, respectively $\overline{C_{(i)}}$, the increasing cumulative sums of coefficients $c_{(1)}$, $c_{(2)}$, ..., $c_{(i)}$, respectively $\overline{c_{(1)}}$, $\overline{c_{(2)}}$, ..., $\overline{c_{(i)}}$.

Lemma 1.

$$\Big(\sum_{i=1}^{2k} c_{(i)} \cdot w_{(i)} > \sum_{i=1}^{2k} \overline{c_{(i)}} \cdot w_{(i)} \Big),\ \forall w \in W(\pi) \ \Leftrightarrow \ \begin{cases} C_{(i)} \leq \overline{C_{(i)}}, i = 1, ..., 2k; \\ \exists i \in 1, ..., 2k : C_{(i)} < \overline{C_{(i)}}. \end{cases} \tag{10}$$

Proof. Demonstration of this lemma (see for instance Fishburn, 1974) goes by rewriting the right hand inequality of Equivalence 9 with the help of the repartition functions $C_{(i)}$ and $\overline{C_{(i)}}$. It readily appears then that the term by term difference of the cumulative sums is conveniently oriented by the right hand conditions of Equivalence 10. □

This concludes the proof of our main result.

Theorem 1. *$\forall a, b \in A$, let $C_{(i)}(a,b)$ represent the increasing cumulative sums of credibilities associated with a given significance ordering of the criteria:*

$$a \, \mathrm{S}_\pi \, b \quad \Leftarrow \quad \begin{cases} C_{(i)}(a,b) \leq \overline{C_{(i)}(a,b)}, i = 1, ..., 2k; \\ \exists i \in 1, ..., 2k : C_{(i)}(a,b) < \overline{C_{(i)}(a,b)}. \end{cases} \tag{11}$$

We observe an ordinal concordant outranking situation between two decision actions a and b as soon as the repartition of credibility on the significance ordering of action a dominates the same of action b.

The preceding result gives us the operational key for testing for the presence of an ordinal concordance situation. Let $L_3 = \{f_\pi, u, t_\pi\}$, where f_π means *rather false than true* with any π-compatible weights w, u means *logically undetermined* and t_π means *rather true than false* with any π-compatible weights w. For each pair of decision actions evaluated in the performance tableau, we may compute such a logical denotation representing truthfulness or falseness of the presence of ordinal concordance in favour of a given outranking situation.

Definition 7. Let π be a significance ordering of the criteria. $\forall a, b \in A$, let $C_{(i)}(a,b)$ and $\overline{C_{(i)}}(a,b)$ denote the corresponding cumulative sums of increasing sums of credibilities associated with the the relation S_π. We define a logical denotation $\|a \text{ S } b\|_\pi$ in L_3 as follows:

$$\|a \text{ S } b\|_\pi = \begin{cases} t_\pi \quad \text{if} & \begin{cases} C_{(i)}(a,b) \leq \overline{C_{(i)}}(a,b), i = 1, ..., 2k \text{ and} \\ \exists i \in 1, ..., 2k : C_{(i)}(a,b) < \overline{C_{(i)}}(a,b); \end{cases} \\ f_\pi \quad \text{if} & \begin{cases} C_{(i)}(a,b) \geq \overline{C_{(i)}}(a,b), i = 1, ..., 2k \text{ and} \\ \exists i \in 1, ..., 2k : C_{(i)}(a,b) > \overline{C_{(i)}}(a,b); \end{cases} \\ u \quad \text{otherwise.} & \end{cases} \tag{12}$$

Coming back to our simple example, we may now apply this test to car models m_4 and m_5 for instance. In Table 2 we have represented the six increasing equi-significance classes we may observe. From Table 1 we may compute the credibilities $c_{(i)}$ (respectively $\overline{c_{(i)}}$) associated with the assertion that model m_4 outranks (respectively does not outrank) m_5 as well as the corresponding cumulative distributions $C_{(i)}$ and $\overline{C_{(i)}}$ as shown in Table 2.

Applying our test, we may notice that indeed $\|m_4 \text{ S } m_5\|_\pi = t_\pi$, i.e. it is true that the assertion "model m_4 outranks model m_5" will be supported by a more or less significant majority of criteria for all π-compatible sets of significance weights.

For information, we may reproduce in Table 3, the complete ordinal outranking relation on A. It is worthwhile noticing that, faithful with the general concordance

Table 2: Assessing the assertion "m_4 S_π m_5"

$\pi_{(i)}$	-Price	-Comfort	-Speed, Design	Speed,Design	Comfort	Price
$c_{(i)}$	0	0	1	1	1	1
$\overline{c_{(i)}}$	1	1	1	1	0	0
$\underline{C_{(i)}}$	0	0	1	2	3	4
$\overline{C_{(i)}}$	1	2	3	4	4	4

Table 3: The ordinal concordance of the pairwise outranking

$\|x \mathrm{~S~} y\|_\pi$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	t_π	u	u	u	u	u
m_2	t_π	-	t_π	f_π	u	f_π	u
m_3	u	t_π	-	u	u	u	u
m_4	t_π	t_π	t_π	-	t_π	t_π	u
m_5	t_π	t_π	t_π	t_π	-	t_π	u
m_6	t_π	t_π	t_π	t_π	t_π	-	t_π
m_7	u	t_π	u	t_π	t_π	t_π	-

principle, the outranking situations a S_π b appearing with value t_π are warranted to be true. Similarly, the situations showing credibility f_π, are warranted to be false. The other situations, appearing with credibility u are to be considered undetermined (see Bisdorff, 2000).

As previously mentioned, model m_6 gives the unique dominant kernel, i.e. a stable and dominant subset, of the $\{f_\pi, u, t_\pi\}$-valued S_π relation. Therefore this decision action represents a robust good choice decision candidate in the sense that it appears to be a rather true than false good choice with all possible π-compatible sets of significance weights (see Bisdorff and Roubens, 2003). Indeed, if we apply the given cardinal significance weights, we obtain in this particular numerical setting that model m_6 is not only among the potential good choices but also, and this might not necessarily always be the case, the most significant one (73%).

Let us now address the robustness issue.

Table 4: The cardinal majority concordance of the outranking of the car models

$r_w(\mathrm{S})$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	.83	.67	.67	.67	.67	.67
m_2	.80	-	.72	.47	.67	.47	.67
m_3	.73	.73	-	.75	.67	.67	.67
m_4	.53	.53	.80	-	.80	.63	.67
m_5	.53	.73	.80	.80	-	.72	.67
m_6	.73	.73	.73	.73	.73	-	.83
m_7	.33	.53	.33	.53	.53	.60	-

4 Analyzing the robustness of global outrankings

Let us suppose that the decision maker has indeed given a precise set w of significance weights. The classical majority concordance will thus deliver a mean weighted outranking relation S_w on A.

In our car selection problem the result is shown in Table 4. We may notice here that for instance $r(m_4 \, \mathrm{S}_w \, m_5) = 80\%$. But we know also from our previuos investigation that $\|m_4 \, \mathrm{S} \, m_5\|_\pi = t_\pi$. The outranking situation is thus confirmed with any π-compatible weight set w.

Going a step further we could imagine a *dream model* that is the cheapest, the most comfortable, very fast and superior designed model, denoted as m_{top}. It is not difficult to see that this model will indeed dominate all the set A with $r(m_{top} \, \mathrm{S} \, x) = 100\%$, i.e. with unanimous concordance $\forall x \in A$. It will naturally also outrank all $x \in A$ in the sense of the ordinal concordance.

4.1 Unanimous concordance

Definition 8. $\forall a, b \in A$ we say that "*a outranks b in the sense of the unanimous concordance principle*", denoted "$a \, \Delta \, b$", if the outranking assertion considered restricted to each individual criterion is *rather true than false.*

We capture once more the potential truthfulness of this dominance assertion with the help of a logical robustness denotation $\|a \, \mathrm{S} \, b\|_\Delta$ taking its values in $L_3 = \{f_\Delta, u, t_\Delta\}$, where f_Δ means *unanimously false*, t_Δ means *unanimously true* and u

means *undetermined* as usual.

$$\forall a, b \in A: \ \|a \,\mathrm{S}\, b\|_\Delta \quad = \quad \begin{cases} t_\Delta & \text{if} \quad \forall g_j \in F: \ r(a \,\mathrm{S}_j\, b) > \frac{1}{2}; \\ f_\Delta & \text{if} \quad \forall g_j \in F: \ r(a \,\mathrm{S}_j\, b) < \frac{1}{2}; \\ u & \text{otherwise.} \end{cases} \tag{13}$$

In our example, neither of the seven models imposes itself on the level of the unanimous concordance principle and the relation Δ remains uniformly undetermined on A.

We are now going to integrate all three outranking relations, i.e. the unanimous, the ordinal and the majority concordance in a common logical framework.

4.2 Integrating unanimous, ordinal and classical majority concordance

Let w represent given numerical significance weights and π the underlying significance preorder. We define the following ordinal sequence (increasing from falsity to truth) of logical robustness degrees: f_Δ means *unanimous concordantly false*, f_π means *ordinal concordantly false with any π-compatible weights*, f_w means *majority concordantly false with weights w*, u means *undetermined*, t_w means *majority concordantly true with weights w*, t_π means *ordinal concordantly true with any π-compatible weights* and t_Δ means *unanimous concordantly true.*

On the basis of a given performance tableau, we may thus evaluate the global outranking relation S on A as follows:

Definition 9. Let $L_7 = \{f_\Delta, f_\pi, f_w, u, t_w, t_\pi, t_\Delta\}$. $\forall a, b \in A$, we define an ordinal robustness denotation $\|a \,\mathrm{S}\, b\| \in L_7$ as follows:

$$\|a \,\mathrm{S}\, b\| \quad = \quad \begin{cases} t_\Delta & \text{if} \quad \|a \,\mathrm{S}\, b\|_\Delta = t_\Delta \ ; \\ t_\pi & \text{if} \quad (\|a \,\mathrm{S}\, b\|_\Delta \neq t_\Delta) \wedge (\|a \,\mathrm{S}\, b\|_\pi = t_\pi) \ ; \\ t_w & \text{if} \quad (\|a \,\mathrm{S}\, b\|_\pi \neq t_\pi) \wedge (\|a \,\mathrm{S}\, b\|_w = t_w) \ ; \\ f_\Delta & \text{if} \quad \|a \,\mathrm{S}\, b\|_\Delta = f_\Delta \ ; \\ f_\pi & \text{if} \quad (\|a \,\mathrm{S}\, b\|_\Delta \neq f_\Delta) \wedge (\|a \,\mathrm{S}\, b\|_\pi = f_\pi) \ ; \\ f_w & \text{if} \quad (\|a \,\mathrm{S}\, b\|_\pi \neq f_\pi) \wedge (\|a \,\mathrm{S}\, b\|_w = f_w) \ ; \\ u & \text{otherwise.} \end{cases} \tag{14}$$

On the seven car models, we obtain for instance the results shown in Table 5. If we apply our methodology for constructing good choices from such an ordinal

Table 5: Robustness of the outranking on the car models

‖S‖	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	t_π	t_w	t_w	t_w	t_w	t_w
m_2	t_π	-	t_π	f_π	t_w	f_π	t_w
m_3	t_w	t_π	-	t_w	t_w	t_w	t_w
m_4	t_π	t_π	t_π	-	t_π	t_π	t_w
m_5	t_w	t_π	t_π	t_π	-	t_π	t_w
m_6	t_π	t_π	t_π	t_π	t_π	-	t_π
m_7	f_w	t_π	f_w	t_π	t_π	t_π	-

valued outranking relation we obtain a single ordinal concordant good choice: model m_6, and four classical majority concordance based good choices: m_1, m_3, m_4 and m_5. The first good choice remains an admissible good choice with any possible π-compatible set of significance weights, whereas the others are more or less dependent on the precise numerical weights given. Similarly, we discover two potentially bad choices: m_2 at the level t_π and m_5 at the level t_w. The first represents therefore a bad choice on the ordinal concordance level.[2]

4.3 Practical applications

In order to illustrate the practical application of the ordinal concordance principle we present two case studies: the first, a classical historical case, well discussed in the literature and a second, very recent real application at the occasion of the EURO 2004 Conference in Rhodes.

4.3.1 Choosing the best postal parcels sorting machine

Let us first reconsider the problem of choosing a postal parcels sorting machine thoroughly discussed in Roy and Bouyssou (1993, pp 501–541).

[2] Conducting a similar analysis with taking into account the veto principle and thresholds given in Vincke (1992), we find that no ordinal concordance is observed anymore. Applying the given numerical significance weights, one gets however that models m_3 and m_4 appear both as potential good choice. Indeed, model m_6 has a weak evaluation on the *comfort* criterion compared to the excellent evaluation of model m_1 for instance, and the same model m_1 is the most expensive one, therefore a veto appears on this criterion in comparison with the prize of model m_7 for instance. Models m_3 and m_4 represent therefore plausible compromises with respect to the numerical significance weights of the criteria. By the way, our example is a nice justification of the usefulness of the veto principle in suitable practical applications.

Table 6: Criteria for selecting a parcel sorting installation

criterion	title	significance weights
g_1	quality of the working place	3/39
g_2	quality of operating environment	2/39
g_3	operating costs	5/39
g_4	throughput	3/39
g_5	ease of operation	3/39
g_6	quality of maintenance	5/39
g_7	ease of installation	2/39
g_8	number of sorting bins	2/39
g_9	investment costs	5/39
g_{10}	bar-code addressing	1/39
g_{11}	service quality	5/39
g_{12}	development stage	3/39

Source: Roy and Bouyssou (1993, p. 527)

We observe a set $A = \{a_1, \ldots, a_9\}$ of 9 potential installations evaluated on the coherent family $F = \{g_1, \ldots, g_{12}\}$ of 12 criteria shown in Table 6. The provided significance weights (see last column) determines the following significance ordering: $w_{10} < w_2 = w_7 = w_8 < w_1 = w_4 = w_5 = w_{12} < w_3 = w_6 = w_9 = w_{11}$. Thus we observe on the proposed family of criteria 4 positive equivalence classes: $\pi_{(5)} = \{g_{10}\}$, $\pi_{(6)} = \{g_2, g_7, g_8\}$, $\pi_{(7)} = \{g_1, g_4, g_5, g_{12}\}$, and $\pi_{(8)} = \{g_3, g_6, g_9, g_{11}\}$ and 4 mirrored negative equivalence classes: $\pi_{(1)} = \{g_3, g_6, g_9, g_{11}\}$, $\pi_{(2)} = \{g_1, g_4, g_5, g_{12}\}$, $\pi_{(3)} = \{g_2, g_7, g_8\}$, $\pi_{(4)} = \{g_{10}\}$.

A previous decision aid analysis has eventually produced a performance tableau of which we show an extract in Table 7. The evaluations on each criterion, except g_9 (*costs of investment* in millions of French francs), are normalized such that $0 \leq g_j(a_i) \leq 100$. If we consider for instance the installations a_1 and a_5, we may deduce the local outranking credibility coefficients $r(a_1 \, \mathrm{S}_j \, a_5)$ shown in Table 7. There is no unanimous concordance in favour of $a_1 \, \mathrm{S} \, a_5$. Indeed we observe on criterion g_4 (*throughput*) a significant negative difference in performance. We may nevertheless observe an ordinal concordance situation $a_1 \; \mathrm{S}_\pi \; a_5$ as distribution $C_{(i)}(a_1, a_5)$ is entirely situated to the right of distribution $\overline{C_{(i)}}(a_1, a_5)$ (see Table 8).

On the complete set of pairwise outrankings of potential installations, we observe the robustness denotation shown in Table 9. We may notice the presence of one unanimous concordance situation $a_4 \Delta a_5$ qualifying the outranking of a_4 over a_5 (see Table 7). Computing from this ordinally valued robust outranking relation all robust

Table 7: Qualifying outranking situation $a_1\,\mathrm{S}_j\,a_5$ and $a_4\,\mathrm{S}_j\,a_5$

g_j	1	2	3	4	5	6	7	8	9	10	11	12
q_j	5	5	5	5	5	10	8	0	1	10	5	10
$g_j(a_1)$	75	69	68	70	82	72	86	74	-15.23	83	76	29
$g_j(a_4)$	73	57	82	90	75	61	93	60	-15.55	83	71	29
$g_j(a_5)$	76	46	55	90	48	46	93	60	-30.68	83	50	14
$r(a_1\,\mathrm{S}_j\,a_5)$	1	1	1	0	1	1	1	1	1	1	1	1
$r(a_4\,\mathrm{S}_j\,a_5)$	1	1	1	1	1	1	1	1	1	1	1	1
$r(\overline{a_1\,\mathrm{S}_j\,a_5})$	0	0	0	1	0	0	0	0	0	0	0	0
$r(\overline{a_4\,\mathrm{S}_j\,a_5})$	0	0	0	0	0	0	0	0	0	0	0	0

Source: Roy and Bouyssou (1993, p. 527)

Table 8: cumulative significance distribution of outranking $a_1\,\mathrm{S}\,a_5$

$\pi_{(i)}$	$\pi_{(1)}$	$\pi_{(2)}$	$\pi_{(3)}$	$\pi_{(4)}$	$\pi_{(5)}$	$\pi_{(6)}$	$\pi_{(7)}$	$\pi_{(8)}$
$C_{(i)}(a_1, a_5)$	0	1	1	1	2	5	8	12
$\overline{C_{(i)}}(a_1, a_5)$	4	7	10	11	11	12	12	12

Table 9: Robustness degrees of outranking situations

$\Vert a_i\,\mathrm{S}\,a_j\Vert$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	-	t_π	t_π	t_π	t_π	t_π	t_π	t_π	t_π
a_2	t_π	-	t_π	t_π	t_π	t_π	t_π	t_π	t_π
a_3	t_π	t_π	-	t_π	t_π	t_π	t_π	t_π	t_π
a_4	t_π	t_π	t_π	-	t_Δ	t_π	t_π	t_π	t_π
a_5	f_π	f_π	f_π	f_π	-	f_π	f_π	f_w	t_π
a_6	t_w	f_w	t_w	t_π	t_π	-	t_w	t_w	t_π
a_7	t_π	t_π	t_w	t_π	t_π	t_π	-	t_π	t_π
a_8	t_w	t_π	t_π	t_π	t_π	t_π	t_π	-	t_π
a_8	f_w	t_π	t_π	t_π	t_π	f_w	f_w	t_π	-

good choices, i.e. minimal dominant sets in the sense of the robust concordance, we obtain that installations a_1, a_2, a_3 and a_4 each one gives a robust good choice at level t_π, whereas the installations a_5 and a_9 give each one a robust bad choice again at level t_π. If we apply in particular the given numerical significance weights (see Table 6), we furthermore obtain that a_1 gives among the four potential good choices the most credible (67%) one whereas among the admissible bad choices it is installation a_5 which gives the most credible (67%) worst one. This result precisely confirms and even formally validates the robustness discussion reported in Roy and Bouyssou (1993, p. 538).

4.3.2 The Euro Best Poster Award 2004: finding a robust consensual ranking

The Programme Committee of the 20th European Conference on Operational Research, Rhodes 2004 has introduced a new type of EURO K conference participation consisting in a daily poster session linked with an oral 30 minutes presentation in front of the poster, a presentation style similar to poster sessions in traditional natural sciences congresses. In order to promote these new discussion presentations,the organizers of the conference proposed a EURO Best Poster Award (EBPA) consisting of a diploma and a prize of 1000 €. Each contributor accepted in the category of the discussion presentations was invited to submit a pdf image of his poster to a five member jury.

The Programme Committee retained the following evaluation criteria: *scientific quality* (sq), *contribution to OR theory and/or practice* (ctp), *originality* (orig) and *presentation quality* (pq) in decreasing order of importance. 13 candidates actually submitted a poster in due time and the five jury members were asked to evaluate the 13 posters on each criteria with the help of an ordinal scale : 0 (very weak) to 10 (excellent) and to propose a global ranking of the posters.

As all five jury members were officially equal in significance, we may consider to be in the presence of a family of $5 \times 4 = 20$ criteria gathered into four equivalence classes listed hereafter in decreasing order of significance: $\pi_{(1)} = \{sq_1, sq_2, sq_3, sq_4, sq_5\}$, $\pi_{(2)} = \{pct_1, pct_2, pct_3, pct_4, pct_5\}$, $\pi_{(3)} = \{orig_1, orig_2, orig_3, orig_4, orig_5\}$ and $\pi_{(4)} = \{pq_1, pq_2, pq_3, pq_4, pq_5\}$.

The cardinal significance weights associated with the four classes of equi-significant criteria were eventually the following: $w_{sq_i} = 4$, $w_{ctp_i} = 3$, $w_{orig_i} = 2$ and $w_{pq_i} = 1$, for $i = 1$ to 4.

The decision problem we are faced with is to aggregate the 20 rankings of the 13 posters on the basis of the given performance tableau. To do so we first computed the credibility index r_w of the global outranking relation S shown in Table 10 using

Table 10: Global outranking of the posters

$r_w(S)$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
p_1	-	.58	.24	.12	.46	.68	.34	.76	.65	.04	.63	.08	.28
p_2	.42	-	.34	.34	.34	.42	.42	.40	.61	.24	.45	.34	.26
p_3	.82	.74	-	.54	.66	.98	.86	.96	.69	.16	.81	.58	.46
p_4	.98	.68	.62	-	.76	.98	.82	.98	.69	.28	.75	.70	.54
p_5	.64	.68	.72	.48	-	1.0	.78	.98	.69	.26	.75	.52	.0
p_6	.54	.58	.10	.10	.34	-	.42	.86	.65	.0	.63	.04	.0
p_7	.68	.72	.32	.46	.30	.86	-	.82	.65	.10	.69	.50	.36
p_8	.50	.60	.16	.20	.30	.66	.40	-	.71	.02	.67	.16	.0
p_9	.43	.49	.35	.35	.41	.49	.37	.49	-	.0	.39	.37	.35
p_{10}	1.0	.80	1.0	.84	1.0	1.0	.90	1.0	.71	-	.81	.88	.80
p_{11}	.71	.61	.37	.29	.29	.43	.39	.59	.69	.0	-	.31	.43
p_{12}	.98	.66	.70	.62	.64	.96	.78	.94	.69	.32	.75	-	.56
p_{13}	.1.0	.76	.70	.60	.80	.80	.70	.96	.69	.48	.81	.64	-

the given significance weights w.

Considering the ordinal character of the criterial scales involved, indifference and preference thresholds were considered to be identically zero, respectively one, on all criteria and no veto thresholds were to be considered. Applying our bipolar ranking approach (see Bisdorff, 1999) to this classical outranking relation gives the following ranking of the posters:

Bipolar ranking of the 13 posters from relation S

Best choice	p_{10}
2nd best choice	p_{13}
3rd best choice	p_4, p_{12}
4th best choice	p_3
5th best choice	p_5
6th best choice	p_7
6th worst choice	p_1
5th worst choice	p_6
4th worst choice	p_8
3rd worst choice	p_{11}
2nd worst choice	p_2
Worst choice	p_9

Poster p_{10} appears majoritarian as the best candidate as it globally outranks all other poster with a comfortable weighted significance of 80%, followed in a second

Table 11: Robust outranking of the posters

‖S‖	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
p_1	-	t_π	f_π	f_π	f_w	t_π	f_π	t_π	t_π	f_π	t_π	f_π	f_π
p_2	f_π	-	f_π	f_π	f_π	f_π	f_π	f_π	t_π	f_π	f_w	f_π	f_π
p_3	t_π	t_π	-	t_w	t_w	t_π	t_π	t_π	t_π	f_π	t_π	t_w	f_w
p_4	t_π	t_π	t_π	-	t_π	t_π	t_π	t_π	t_π	f_w	t_π	t_π	t_π
p_5	t_π	t_π	t_π	f_w	-	t_Δ	t_π	t_π	t_π	f_π	t_π	t_w	f_Δ
p_6	t_w	t_π	f_π	f_π	f_π	-	f_π	t_π	t_π	f_Δ	t_π	f_π	f_Δ
p_7	t_π	t_π	f_π	f_w	f_π	t_π	-	t_π	t_π	f_π	t_π	u	f_π
p_8	u	t_π	f_π	f_π	f_π	t_π	f_π	-	t_π	f_π	t_π	f_π	f_Δ
p_9	f_π	f_π	f_π	f_π	f_π	f_π	f_π	f_π	-	f_Δ	f_π	f_π	f_π
p_{10}	t_Δ	t_π	t_Δ	t_π	t_Δ	t_Δ	t_π	t_Δ	t_π	-	t_π	t_π	t_π
p_{11}	t_π	t_π	f_π	f_π	f_π	f_π	f_π	t_π	t_π	f_Δ	-	f_π	f_π
p_{12}	t_π	t_π	t_π	t_w	t_π	t_π	t_π	t_π	t_π	f_w	t_π	-	t_π
p_{13}	t_Δ	t_π	t_π	t_w	t_π	t_π	t_π	t_π	t_π	f_w	t_π	t_π	-

position by poster p_{13} and posters p_4 and p_{12} ex eaquo in a third position. On the other side, poster p_9 appears to be the least appreciated by the judges (overall significance: 60%), preceded by poster p_2 in the second worst position. But is this precise consensual ordering not an artifact induced by our more or less arbitrarily chosen cardinal importance weights: 4, 3, 2, 1 ? To check this point, we compute the robustness degrees of the previous outranking relation as shown in Table 11. Directly applying the same bipolar ranking approach to the ordinal valued ‖S‖ outranking relation, we obtain the following ordering:

Bipolar ranking of the 13 posters from relation ‖S‖

Best choice	p_{10}
2nd best choice	p_4
3rd best choice	p_{12}, p_{13}
4th best choice	p_5
5th best choice	p_3
6th best choice	p_7
6th worst choice	p_7
5th worst choice	p_1
4th worst choice	p_6
3rd worst choice	p_8, p_{11}
2nd worst choice	p_2
Worst choice	p_9

Previous results get well confirmed on the whole. Indeed with a robustness degree of

t_π, i.e. rather true than false with any π-compatible weights, poster p_{10} is confirmed in the first[3] and poster p_9 in the last position [4].

Attributing the EBPA 2004 to poster p_{10} was therefore indeed independent of the choice of any precise numerical significance weights verifying the significance ordering of the four criteria as imposed by the Programme Committee.

5 Conclusion

In this paper we have presented a formal approach for assessing binary outranking situations on the basis of a performance tableau involving criteria of solely ordinal significance. The concept of ordinal concordance has been introduced and a formal testing procedure based on distributional dominance is developed. Thus we solve a major practical problem concerning the precise numerical knowledge of the individual significance weights that is required by the classical majority concordance principle as implemented for instance in the Electre methods. Applicability of the concordance based aggregation of preference is extended to the case where only ordinal significance of the criteria is available. Furthermore, even if precise numerical significance is available, we provide a robustness analysis of the observed preferences by integrating unanimous, ordinal and majority based concordance in a same logical framework.

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[3]Poster p_{10}, which obtained the EBPA 2004, was submitted by Federica RICCA, Bruno SIMEONE and Isabella LARI on *Political Districting via Weighted Voronoï Regions* from the University of Rome "La Sapienza".

[4]It is worthwhile noticing that our bipolar ranking method was not designed to be necessarily stable with respect to the above robustness analysis. And indeed, we may notice a slight order reversal concerning respective positions of posters p_4 and p_{13}. But otherwise there appears no major divergence between both orderings.

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