On Clustering the Criteria in an Outranking Based Decision Aid Approach

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Abstract. In this paper we discuss the clustering of the set of criteria in a multicriteria decision analysis. Our approach is based on a generalisation of Kendall's rank correlation resulting in the definition of a bipolar ordinal correlation index. A factorial decomposition of this index allows to compute the principal inertia planes of the criteria correlations. The same ordinal correlation index, modelling a symmetric bipolar-valued similarity digraph, allows us to compute a criteria clustering from its maximal cliques.

Introduction

The PROMETHEE authors [1,2] consider very accurately that one of the methodological requisites for an appropriate Multicriteria Decision Aid (MCDA) method is the necessity to provide information on the *conflicting nature* of the criteria. The classical Electre methods [3,4] as well as the recent Rubis best choice method [5] do not provide any such information. In this paper we therefore present several tools that, similar in their operational purpose to the PROMETHEE GAIA plane [2], help illustrating concordance and/or discordance of the criteria with respect to the preferential judgments they show on the given set of decision alternatives. The following example will illustrate our discussion all along the paper.

Example 1 (The Ronda choice decision problem). A family, staying during their holidays in Ronda (Andalusia), is planning the next day's activity. The alternatives shown in Table 1 are considered as potential action. The family members agree to measure their preferences with respect to a set of seven criteria such as the time to attend the place (to be minimised), the required physical investment, the expected quality of the food, touristic interest, relaxation, sun fun & more, ... (see Table 2). The common evaluation of the performances of the nine alternatives on all the criteria results in the performance table shown in Table 3. All performances on the qualitative criteria are marked on a same ordinal scale going from 0 (lowest) to 10 (highest). On the quantitative *Distance* criterion (to be minimized), the required travel time to go to and return from the activity is marked in negative minutes. In order to model only effective preferences, an indifference threshold of 1 point and a preference threshold of 2 points is put

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Table 1. Ronda example: The set of alternatives

Identifier	Name	Comment
ant	Antequerra	An afternoon excursion to Antequerra and surroundings.
ard	Ardales	An afternoon excursion to Ardales and El Chorro.
be	beach	Sun, fun and more.
crd	Cordoba	A whole day visit by car to Cordoba.
dn	fa niente	Doing nothing.
lw	long walk	A whole day hiking.
mal	Malaga	A whole day visit by car to Malaga.
sev	Sevilla	A whole day visit by car to Sevilla.
sw	short walk	Less than a half day hiking.

Table 2. Ronda example: The set of criteria

Identifier	Name	Comment
cult	*Cultural Interest*	Andalusian heritage.
dis	*Distance*	Minutes by car to go to and come back from the activity.
food	*Food*	Quality of the expected food opportunities.
sun	*Sun, Fun, & more*	No comment.
phy	*Physical Investment*	Contribution to physical health care.
rel	*Relaxation*	Anti-stress support.
tour	*Tourist Attraction*	How many stars in the guide ?

Table 3. Ronda example: The performance table

Criteria	ant	ard	be	crd	dn	lw	mal	sev	sw
cult	7.0	3.0	0.0	10.0	0.0	0.0	5.0	10.0	0.0
dis	-120.0	-100.0	-30.0	-360.0	0.0	-90.0	-240.0	-240.0	0.0
phy	3.0	7.0	0.0	5.0	0.0	10.0	5.0	5.0	5.0
rel	1.0	5.0	8.0	3.0	10.0	5.0	3.0	3.0	6.0
food	8.0	10.0	4.0	8.0	10.0	1.0	8.0	10.0	1.0
sun	0.0	3.0	10.0	3.0	1.0	3.0	8.0	5.0	5.0
tour	5.0	7.0	3.0	10.0	0.0	8.0	10.0	10.0	5.0

on the qualitative performance measures. On the *Distance* criterion, an indifference threshold of 20 min, and a preference threshold of 45 min. is considered. Furthermore, a difference of more than two hours to attend the activity's place is considered to raise a veto (see Table 4).

The individual criteria each reflect one or the other member's preferential point of view. Therefore they are judged equi-significant for the best action to be eventually chosen.

How do the criteria express their preferential view point on the set of alternatives? For instance the *Tourist Attraction* criterion appears to be in its preferential judgments somehow positively correlated with both the *Cultural Interest* and the *Food* criteria. It is also apparent that the *Distance* criterion is somehow negatively correlated to these latter criteria. How can we explore and illustrate these intuitions?

In a given MCDA, where a certain set of criteria is used for solving a given decision problem, it is generally worthwhile analysing to what extent the

Table 4. Ronda example: Preference discrimination thresholds

Criterion	Thresholds indifference	preference	veto
cult	1pt	2pts	-
dis	20min.	45min.	121min.
food	1pt	2pts	-
sun	1pt	2pts	-
phy	1pt	2pts	-
rel	1pt	2pts	-
tour	1pt	2pts	-

criteria vary in their relational judgments concerning the pairwise comparison of performances of the alternatives. Illustrating such similarities and dissimilarities between criteria judgments is indeed the very purpose of this paper. First, we present a bipolar-valued ordinal criteria correlation index, generalising Kendall's τ [6], and illustrating the preferential distance between the criterial judgments. In a second section we show how to decompose this correlation index into its principal components. In a third section, following an earlier work of ours [11], we propose a credibility level indexed clustering of the criteria based on the extraction of maximal bipolar-valued cliques observed in the associated criteria similarity digraph.

1 A Bipolar-Valued Ordinal Criteria Correlation Index

Let us introduce our notations. We consider a finite set A of n alternatives and denote by x and y any two alternatives. We consider also a set F of outranking criteria [4] denoted by variables i or j, with $k = 0, 1, ...$ discrimination thresholds. The performance of an alternative x on criterion i is denoted by x_i.

Example 2. The four discrimination thresholds we may observe on each criterion i for instance in the Rubis choice method [5] are: – "weak preference"[1] wp_i ($0 < wp_i$), – "preference" p_i ($wp_i \leqslant p_i$), – "weak veto" wv_i ($p_i < wv_i$), and – "veto" v_i ($wv_i \leqslant v_i$). Each difference $(x_i - y_i)$ may thus be classified into one and only one of the following nine cases:

($\ggg$) *"veto against $x \leqslant y$"* $\Leftrightarrow$ $v_i \leqslant (x_i - y_i)$
($\gg$) *"weak veto against $x \leqslant y$"* $\Leftrightarrow$ $wv_i \leqslant (x_i - y_i) < v_i$
($>$) *"x better than y"* $\Leftrightarrow$ $p \leqslant (x_i - y_i)$
($\geqslant$) *"x better than or equal y"* $\Leftrightarrow$ $wp_i \leqslant (x_i - y_i) < p_i$
($=$) *"x indifferent to y"* $\Leftrightarrow$ $-wp_i < (x_i - y_i) < wp_i$
($\leqslant$) *"x worse than or indifferent to y"* $\Leftrightarrow$ $-p_i < (x_i - y_i) \leqslant -wp_i$
($<$) *"x worse than y"* $\Leftrightarrow$ $-wp_i < (x_i - y_i) \leqslant -p_i$
($\ll$) *"weak veto against $x \geqslant y$"* $\Leftrightarrow$ $-v_i < (x_i - y_i) \leqslant -wp_i$
($\lll$) *"veto against $x \geqslant y$"* $\Leftrightarrow$ $(x_i - y_i) \leqslant -v_i$

[1] In some cases it may be useful to replace the weak preference threshold, defining an open indifference interval on the criterion scale, with an indifference threshold $0 \leqslant h$ defining a closed indifference interval and leaving open the weak preference interval (see [5]).

In general, let us consider on each criterion i, supporting a set of discrimination thresholds p_r $(r = 1,..,k)$ such that $0 < p_1 \leqslant ... \leqslant p_k$, the Kendall vector (see [7]) gathering the classification of all possible differences $(x_i - y_i)$ into one of the following $2k + 1$ cases:

$$(x_i - y_i) \in \begin{cases} (>_k) & \text{if} \quad p_k \leqslant (x_i - y_i) \\ (>_r) & \text{if} \quad p_r \leqslant (x_i - y_i) < p_{r+1}, \text{for } r = 1, ...k-1 \\ (=) & \text{if} \quad -p_1 < (x_i - y_i) < p_1 \\ (<_r) & \text{if} \quad -p_{r+1} < (x_i - y_i) \leqslant -p_r, \text{for } r = 1, ...k-1 \\ (<_k) & \text{if} \quad (x_i - y_i) \leqslant -p_k \end{cases} \tag{1}$$

Comparing the preferential view point of two criteria i and j, we say that x and y are *concordantly* (resp. *discordantly*) compared if $(x_i - y_i)$ and $(x_j - y_j)$ are classified into the same category (resp. different categories) on both criteria. This is the case if position (i, j) in both Kendall vectors is of the same (resp. different) value. There are $n(n-1)$ distinct ordered pairs of performances and each pair (x, y) is thus either concordantly or discordantly classified. Please notice that we may well compare two criteria with a different number of discrimination thresholds. The only semiotic restriction we require here is that the preferential meanings of the k thresholds are the same for all criteria in the given family F. Denoting by S_{ij} the number c_{ij} of concordantly classified minus the number d_{ij} of discordantly classified ordered pairs, the *ordinal criteria correlation index* $\widetilde{T}$ is defined on $F \times F$ as

$$\tilde{T}(i, j) = \frac{c_{ij} - d_{ij}}{c_{ij} + d_{ij}} = \frac{S_{ij}}{n(n-1)}. \tag{2}$$

Property 1. The ordinal criteria correlation index $\tilde{T}$ is symmetrically valued in the rational bipolar credibility domain $[-1, 1]$ (see [8,5]).

Proof. If all pairs of alternatives are concordantly (discordantly) classified by both criteria, $d_{ij} = 0$ (resp. $c_{ij} = 0$) and $\widetilde{T}(i, j) = 1.0$ (resp. -1.0). If $\widetilde{T}(i, j) > 0$ (resp. < 0) both criteria are more *similar than dissimilar* (resp. *dissimilar than similar*) in their preferential judgments. When $\widetilde{T}(i, j) = 0.0$, no conclusion can be drawn. The linear structure of the criterion scale and the relational coherence of the discrimination thresholds imply that a performance difference $(x_i - y_i)$ is classified in one and only one case. Furthermore, the case of $(x_i - y_i)$ corresponds bijectively to a unique symmetric case classifying the reversed difference $(y_i - x_i)$. Hence, the pair (x, y) is concordantly classified by criteria i and j if and only if the symmetric pair (y, x) is concordantly classified by the same two criteria. □

Property 2. If i and j are two perfectly discriminating criteria, i.e. they admit a single preference threshold $p_1 = \epsilon$, and we don't observe ties in the performance table then $\widetilde{T}(i, j)$ is identical with the classical τ of Kendall [6].

Proof. In this case, both the Kendall vectors of criteria i and j contain only the two possible cases: – case $(>_1)$: $(x_i - y_i) \geqslant \epsilon$, and – case $(<_1)$: $(x_i - y_i) \leqslant \epsilon$. Denoting by p_{ij} the number of pairs (x, y) in $A \times A$ such that conjointly $(x_i - y_i) \geqslant \epsilon$ and $(x_j - y_j) \geqslant \epsilon$ we obtain indeed

$$\widetilde{T}(i,j) = (2 \times \frac{2p_{ij}}{n(n-1)}) - 1, \quad \forall (i,j) \in F \times F, \tag{3}$$

i.e. Kendall's original τ definition (see [6]). □

It is worthwhile noticing that the classical problem for applying Kendall's τ to a situation with ties is here coherently resolved. Indeed, Equation 2 generalises Kendall's rank correlation index to any family of homogeneous semiorders (see [9] Chapter 3).

Example 3 (The Ronda decision problem – continued). Computing our ordinal criteria correlation index $\widetilde{T}$ (see Equation 2) on the set of seven criteria we obtain the results shown in Table 5. As initially suspected, on the one hand, we observe here that the performances on the criteria *Cultural interest* and *Tourist Attraction*, and *Physical Investment* lead to positively correlated preferential judgments ($\widetilde{T}(\text{cult}, \text{tour}) = +0.28$ and $\widetilde{T}(\text{phy}, \text{tour}) = +0.33$). On the other hand, the performances observed on criteria *Distance* and *Cultural Interest* or *Tourist Attraction* lead to nearly completely opposed preferential statements ($\widetilde{T}(\text{dis}, \text{tour}) = -0.92$ and $\widetilde{T}(\text{dis}, \text{cult}) = -0.89$).

As these couples of concordant and/or discordant criteria play an essential role in the actual difficulty of the decision making process, we look for a systematic graphical illustration of the ordinal criteria correlation index.

Table 5. Ronda example: The ordinal criteria correlation table

$\widetilde{T}$	dis	phy	rel	food	sun	tour
cult	−0.89	−0.17	−0.81	+0.00	−0.39	+0.28
dis		−0.72	−0.08	−0.67	−0.39	−0.92
phy			−0.17	−0.39	−0.28	+0.33
rel				−0.25	−0.17	−0.53
food					−0.56	−0.17
sun						−0.03

2 Principal Component Analysis of the Criteria Correlation

A most suitable tool is given by the classical Principal Component Analysis – PCA [10]. We may uncover the principal components of $\widetilde{T}$ by computing the eigen-vectors of its associated covariance. Projecting the criteria points in the covariance eigen-space along the principal coordinates explaining the largest part of the total variance reveals the major agreements and oppositions between the preferential judgments as expressed by the criteria on the given set of alternatives.

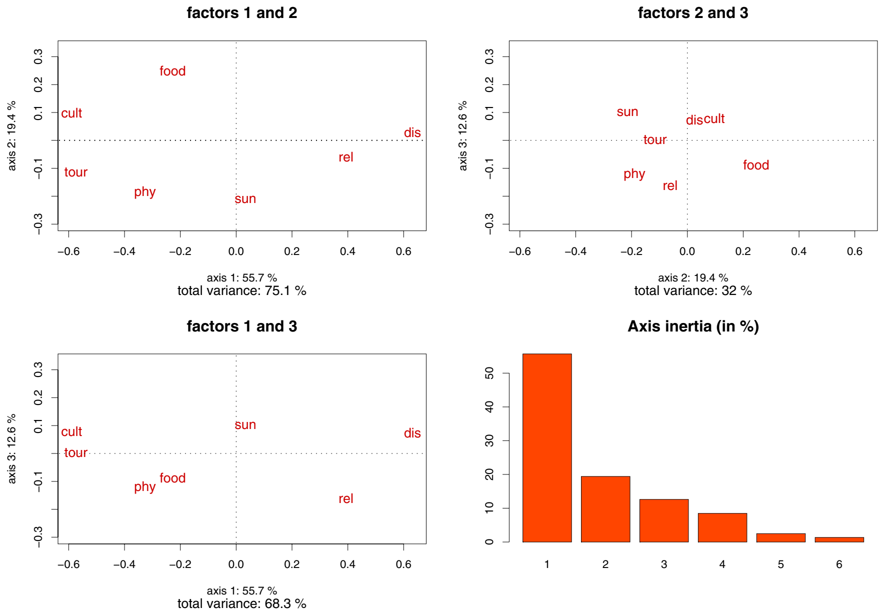

Fig. 1. Ronda example: Results of the PCA

Example 4 (The Ronda decision problem – continued). Such PCA results, computed from the $\widetilde{T}$ index observed in the Ronda example, are shown in Figure 1. As expected, the first and largely prominent opposition – gathering 55.7% of the total variance – is observed between, on the one hand, both criteria *Cultural Interest* and *Tourist Attraction*, and, on the other hand, criterion *Distance* and, to a lesser extent, criterion *Relaxation*. The second factorial axis – already much less prominent (only 19.5% of total variance) – shows an opposition between, on the one hand, the *Food* criterion and, on the other hand, both the *Sun, Fun & more* and the *Physical Investment* criteria. It is furthermore worthwhile noticing that all seven criteria appear in a more or less elliptic layout in the main principal plane (gathering 75.1% of all variance) and thereby indicate that each one owns a specific preferential judgment behaviour, somehow different from all the others.

The PCA of $\widetilde{T}$ is much like the well known PROMETHEE Gaia approach [2]. Main difference is that the Gaia PCA is realized on the covariance of the rows – describing the alternatives – of the single net flows matrix (see [2]). Recall that the single net flow for alternative x on criteria i is the normalized difference between the number of times x is preferred to the other alternatives minus the number of times the other alternatives are preferred to x. The Gaia plane therefore shows the projection of the alternatives in the plane of the two most prominent principal axes. The criteria are there only indirectly represented as supplementary points, the unit vectors of the coordinate axis representing each criteria. Practical experiments have shown that very similar result to ours would however appear when realizing a PCA on the covariance, not of the rows, but of the columns – describing the criteria – of the single net flows matrix. Main advantage of our $\widetilde{T}$ measure is, nonetheless, that the distance between the criteria's preferential judgments is not computed from a compound preference situation, but takes into account the indifference situation as well as all k discriminated preference levels a criterion may, the case given, attach to a given performance difference on all pairs of alternatives.

If the PCA of the criteria correlation index $\widetilde{T}$ reveals very convincingly the most prominently opposed criteria, the projection of the criteria into the main principal planes also illustrates quite well the potential proximities between criteria (see the position of criteria *Cultural Interest* and *Tourist Attraction* for instance in Figure 1). In order to qualify the credibility of such proximities, we finally propose a bipolar-valued clustering based again on the ordinal criteria correlation index $\widetilde{T}$.

3 Bipolar-Valued Clustering from the Criteria Correlation Index $\widetilde{T}$

For this last approach, we make use of Property 1 which tells us that the index $\widetilde{T}$ represents a bipolar-valued characteristic denotation of the propositional statement "*criteria i and j express similar preferential statements on A*". We consider indeed this statement to be more or less validated if both criteria are concordant on a *majority* of pairwise comparisons and discordant on a minority ones. In this sense, $\widetilde{T}$ is characterising a bipolar-valued *similarity* graph, we denote by $\widetilde{S}(F,\widetilde{T})$ or $\widetilde{S}$ for short. Following from the logical denotation of the bipolar valuation, we say that there is an arc between i and j if $\widetilde{T}(i,j) > 0$ (see [8]). Similarly, a clique C in $\widetilde{S}$ is a subset of criteria such that for all i and j in C, we have $\widetilde{T}(i,j) \geqslant 0$. [2]

In general, we may associate a crisp graph $S(F,T)$ with $\widetilde{S}$, where $T = \{(i,j)|\widetilde{T}(i,) > 0\}$. All properties of S are canonically transferred to $\widetilde{S}$. For instance, S is a symmetric digraph (see Property 1), so is $\widetilde{S}$.

Example 5 (The Ronda decision problem – continued). The criteria similarity graph in the Ronda example contains only three edges: – between *Physical Investment* and *Tourist Attraction* ($\widetilde{T}$(phy,tour) = 0.33), – between *Tourist Attraction* and *Cultural Interest* ($\widetilde{T}$(tour,cult) = 0.28), and – the weak (or potential) similarity between criteria *Food* and *Cultural Interest* ($\widetilde{T}$(food,cult) = 0.0). Notice that the similarity relation is not transitive (a fact easily explainable from Figure 1).

[2] We admit here a weak notion of a bipolar-valued clique by including possibly indeterminate similarity situations. A strict bipolar-valued clique concept would require a strictly positive valuation.

What we are looking for are maximal cliques, i.e. subsets C of criteria which verify both the following properties:

1. *Internal stability*: all criteria in C are similar, i.e. the subgraph $(C, \widetilde{T}_{|C})$ is a clique;
2. *External stability*: if a criteria i is not in C, there must exist a criteria j in C such that $\widetilde{T}(i,j) < 0$ and $\widetilde{T}(j,i) < 0$.

For any $C \in F$, we denote by $\Delta^{int}(C)$ (resp. $\Delta^{ext}(C)$) its credibility of being internally (resp. externally) stable:

$$\Delta^{int}(C) = \begin{cases} 1.0 & \text{if} \quad |C| = 1, \\ \min_{i \in C} \min_{j \in C}^{j \neq i} \big(\widetilde{T}(i,j)\big) & \text{otherwise.} \end{cases} \tag{4}$$

$$\Delta^{ext}(C) = \begin{cases} 1.0 & \text{if} \quad C = F, \\ \min_{i \in F}^{i \notin C} \max_{j \in C} \big(- \widetilde{T}(i,j)\big) & \text{otherwise.} \end{cases} \tag{5}$$

Property 3. A subset C of criteria is a maximal clique of the similarity graph $\widetilde{S} \equiv (F, \widetilde{T})$ if and only if both $\Delta^{int}(C) \geqslant 0$ and $\Delta^{ext}(C) > 0$.

Proof. Condition $\Delta^{int}(C) \geqslant 0$ directly implies that $(C, \widetilde{T}_{|C})$ is a clique and condition $\Delta^{ext}(C) > 0$ implies that, for any criterion i not in C, there exists at least one criterion j in C such that $\widetilde{T}(i,j) < 0$. □

Computing maximal cliques in a graph is equivalent to the problem of computing maximal independent sets in the dual graph. These problems are in theory algorithmically difficult [12]. Considering however the very low dimension of the set of criteria in a common MCDA problem, there is no operational difficulty here for the decision aid practice. The credibility level $\min(\Delta^{ext}, \Delta^{int})$ of the resulting maximal cliques may eventually lead to a bipolar-valued clustering of the family of criteria (see [11]).

Example 6 (The Ronda decision problem – continued). The clustering results are shown in Table 6.

The most validated maximal cliques (at credibility level 58.33%[3]) are the pairs (*Physical Investment*, *Tourist Attraction*) and (*Tourist Attraction*, *Cultural Interest*). At level 54.17%, both the criteria *Distance* and *Relaxation* are singleton maximal cliques, followed at level 51.39% by the criterion *Sun, Fun & more.* Finally, a potential maximal clique is the pair (*Cultural Interest*, *Food*). The credibility level indexed clustering results are shown in Figure 2.

With non-redundant and preferentially independent criteria, we may expect in general very small maximal cliques and singletons. Monte Carlo experiments with random performance tableaux confirm indeed this sparsity of the criteria clustering in normal MCDA problems.

[3] The credibility levels are expressed as $(\min(\Delta^{ext}, \Delta^{int}) + 1.0)/2.0$ in the $[0, 1]$ interval.

Table 6. Ronda example: Clustering the criteria

Maximal cliques	credibility level (in%[3])	stability external	stability internal
{phy,tour}	58.33	+0.167	+0.333
{tour,cult}	58.33	+0.167	+0.278
{dis}	54.17	+0.083	+1.00
{rel}	54.17	+0.083	+1.00
{sun}	51.39	+0.028	+1.00
{cult,food}	50.00	+0.167	0.0

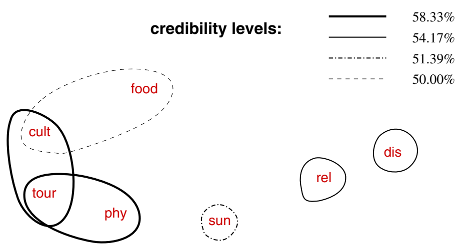

Fig. 2. Ronda example: The bipolar-valued criteria clusters

Conclusion

Despite the obvious importance of the methodological requisite for a suitable MCDA approach to offer tools for illustrating preferential agreements and/or oppositions between the criteria, no specific formal methodological contribution, apart from the PROMETHEE Gaia plane, has been made in the general context of the outranking based MCDA methods. This paper fills this gap with a generalisation of Kendall's rank correlation τ measure to the pairwise comparison of the preferential judgements the criteria apply to a given set of alternatives. This new ordinal criteria correlation index may be used, on the one hand, for graphically illustrating oppositions and agreements between criteria with the help of a PCA similar to the Gaia approach. On the other hand, the same ordinal correlation index may also be used for extracting in a decreasing level of credibility the maximal cliques from a bipolar-valued criteria similarity graph.

References

1. Brans, J.P.: L'ingénierie de la dcision: Élaboration d'instruments d'aide à la décision. La méthode PROMETHEE. In: Nadeau, R., Landry, M. (eds.) L'aide à la décision: Nature, Instruments et Perspectives d'avenir, pp. 183–213. Presses de l'Université Laval, Québec (1982)
2. Brans, J.P., Mareschal, B.: The Promcalc and Gaia decision-support system for multicriteria decision aid. Dec. Sup. Sys. 12, 297–310 (1994)

3. Roy, B.: Méthodologie Multicritère d'Aide à la Décision. Economica Paris (1985)
4. Roy, B., Bouyssou, D.: Aide Multicritére à la Décision: Méthodes et Cas. Economica Paris (1993)
5. Bidorff, R., Meyer, P., Roubens, M.: RUBIS: a bipolar-valued outranking method for the choice problem. 4OR, pp. 1–27. Springer, Heidelberg (2008)
6. Kendall, M.G.: Rank Correlation Methods. Hafner Publishing Co., New York (1955)
7. Degenne, A.: Techniques ordinales en analyse des données statistique. Hachette, Paris (1972)
8. Bisdorff, R.: Logical Foundation of Multicriteria Preference Aggregation. In: Bouyssou, D., et al. (eds.) Essay in Aiding Decisions with Multiple Criteria, pp. 379–403. Kluwer Academic Publishers, Dordrecht (2002)
9. Bouyssou, D., Marchant, Th., Pirlot, M., Tsoukias, A., Vincke, Ph.: Evaluation and Decision Models with Multiple Criteria. Springer, New York (2006)
10. Benzecri, J.P.: L'analyse des données – Tome 2: L'Analyse des correspondances. Dunod Paris (1976)
11. Bidorff, R.: Electre-like clustering from a pairwise fuzzy proximity index. Eur. Jour. Oper. Res. 138, 320–331 (2002)
12. Bisdorff, R.: On enumerating the kernels in a bipolar-valued outranking digraph. Annales du Lamsade 6, 1–38 (2006)