On weakly ordering by choosing from valued pairwise outranking situations

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Content

Sample performance tableau

Let $X = \{a_1, ..., a_7\}$ be seven potential decision actions evaluated on three cost criteria (g_1, g_4, g_5) of equi-significance $1/6$ and two benefit criteria (g_2, g_3) of equi-signifiance $1/4$. The given performance tableau is shown below.

Objectives	Costs			Benefits	
Criteria	$g_1(\downarrow)$	$g_4(\downarrow)$	$g_5(\downarrow)$	$g_2(\uparrow)$	$g_3(\uparrow)$
weights×12	2.0	2.0	2.0	3.0	3.0
indifference	3.41	4.91	-	-	2.32
preference	6.31	8.31	-	-	5.06
veto	60.17	67.75	-	-	48.24
a_1	22.49	36.84	7	8	43.44
a_2	16.18	19.21	2	8	19.35
a_3	29.41	54.43	3	4	33.37
a_4	82.66	86.96	8	6	48.50
a_5	47.77	82.27	7	7	81.61
a_6	32.50	16.56	6	8	34.06
a_7	35.91	27.52	2	1	50.82

Sample outranking relation

The resulting bipolar outranking relation S is shown below.

Table: r-valued bipolar outranking relation

$r(S) \times 12$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	–	**0**	+8	+12	+6	+4	−2
a_2	+6	–	+6	+12	**0**	+6	+6
a_3	−8	−6	–	**0**	−12	+2	−2
a_4	−12	−12	**0**	–	−8	−12	**0**
a_5	−2	**0**	+12	+12	–	−6	**0**
a_6	+2	+4	+8	+12	+6	–	+2
a_7	+2	−2	+2	+6	**0**	+2	–

1. a_6 is a Condorcet winner,
2. a_2 is a weak Condorcet winner,
3. a_4 is a weak Condorcet looser.

Ranking by Rubis best and worst choosing

- Let X_1 be the set X of potential decision actions we wish to rank.
- While the remaining set X_i $(i = 1, 2, \ldots)$ of decision actions to be ranked is not empty, we extract from X_i the best (B_i), respectively worst (W_i) Rubis choice recommendations and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.
- Both iterations determine, hence, two – usually slightly different – opposite weak orderings on X:
  1. a ranking-y-best-choosing order and,
  2. a ranking-by-worst-rejecting order.

Fusion of best and worst choice rankings

Ranking by recursively choosing:

1st Best Choice ['a02','a05']
- 2nd Best Choice ['a06']
- 3rd Best Choice ['a07']
- 4th Best Choice ['a01']
- 5th Best Choice ['a03','a04']

Ranking by recursively rejecting:

Last Choice ['a03','a04']
- 2nd Last Choice ['a05','a07']
- 3rd Last Choice ['a01']
- 4th Last Choice ['a06']
- 5th Last Choice ['a02']

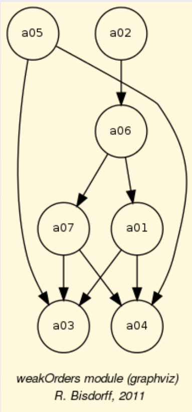
We may fuse both rankings, the first and the converse of the second, with the help of the epistemic conjunction operator ($\oslash$) to make apparent a valued relation R which represents a weakly complete and transitive closure of the given bipolar valued outranking.

Let ϕ and ψ be two logical formulas:

$$\phi \oslash \psi = \begin{cases} (\phi \wedge \psi) & \text{if } \ (\phi \wedge \psi) \text{ is true;} \\ (\phi \vee \psi) & \text{if } \ (\neg\phi \wedge \neg\psi) \text{ is true;} \\ \text{Indeterminate} & \text{otherwise.} \end{cases}$$

Table: Weakly complete transitive closure of S

r(R)	a_2	a_5	a_6	a_1	a_7	a_3	a_4
a_2	–	**0**	+6	+6	+6	+6	+12
a_5	**0**	–	**0**	**0**	**0**	+12	+12
a_6	−4	**0**	–	+2	+2	+8	+12
a_1	0	**0**	−4	–	**0**	+8	+12
a_7	−2	**0**	−2	**0**	–	+2	+6
a_3	−6	−12	−2	−8	−2	–	**0**
a_4	−12	−8	−12	−12	0	**0**	–



Notice the contrasted ranks of action a_5 (first best as well as second last), indicating a lack of comparability, which becomes apparent in the conjunctive epistemic fusion R of both weak orderings shown in the Table above and illustrated in the corresponding Hasse diagram.

Bipolar characteristic function r

- $X = \{x, y, z, \ldots\}$ is a finite set of m decision alternatives;
- We define a binary relation R on X with the help of a bipolar characteristic function r taking values in the rational interval $[-1.0; 1.0]$.
- **Bipolar semantics**: For any pair $(x, y) \in X^2$,
  1. $r(x\,R\,y) \ = \ +1.0$ means $x\,R\,y$ valid for sure,
  2. $r(x\,R\,y) \ > \ 0.0$ means $x\,R\,y$ more or less valid,
  3. $r(x\,R\,y) \ = \ 0.0$ means both $x\,R\,y$ and $x \not R\, y$ indeterminate,
  4. $r(x\,R\,y) \ < \ 0.0$ means $x \not R\, y$ more or less valid,
  5. $r(x\,R\,y) \ = \ -1.0$ means $x \not R\, y$ valid for sure.
- **Boolean operations**: Let ϕ and ψ be two relational propositions.
  1. $r(\neg\phi) \ = \ -r(\phi)$.
  2. $r(\phi \vee \psi) \ = \ \max\big(r(\phi), r(\psi)\big)$,
  3. $r(\phi \wedge \psi) \ = \ \min\big(r(\phi), r(\psi)\big)$.

Weakly complete binary relations

Let R be an r-valued binary relation defined on X.

Definition

We say that R is weakly complete on X if, for all $(x, y) \in X^2$, **either** $r(x \, \mathrm{R} \, y) \geqslant 0.0$ **or** $r(y \, \mathrm{R} \, x) \geqslant 0.0$.

Examples

1. Marginal semi-orders observed on each criterion,
2. Weighted condordance relations,
3. Polarised outranking relations,
4. Ranking-by-choosing results,
5. Weak and linear orderings.

Universal properties

Let $\mathcal{R}$ denote the set of all possible weakly complete relations definable on X.

Property ($\mathcal{R}$-internal operations)

1. *The convex combination of any finite set of such weakly complete relations remains a weakly complete relation.*
2. *The disjunctive combination of any finite set of such weakly complete relations remains a weakly complete relation.*
3. *The epistemic-conjunctive (resp. -disjunctive) combination of any finite set of such weakly complete relations remains a weakly complete relation.*

Examples: Concordance of linear-, weak- or semi-orders, bipolar outranking (concordance-discordance) relations.

Useful properties

We say that a binary relation $R \in \mathcal{R}$ verifies the *coduality principle* ($> \equiv \not\leq$), if the converse of its negation equals its asymetric part : $\min\left(r(x \, \mathrm{R} \, y), -r(y \, \mathrm{R} \, x)\right) \; = \; -r(y \, \mathrm{R} \, x)$. Let $\mathcal{R}^{cd}$ denote the set of all possible relations $R \in \mathcal{R}$ that verify the coduality principle.

Property (Coduality principle)

The convex and epistemic-disjunctive (resp. -conjunctive) combinations of a finite set of relations in $\mathcal{R}^{cd}$ verify again the coduality principle.

Examples: Marginal linear-, weak- and semi-orders; concordance and bipolar outranking relations; all, verify the coduality principle.

Pragmatic principles of the RUBIS choice

$\mathcal{P}_1$: Elimination for well motivated reasons:
Each eliminated alternative has to be outranked by (resp. is outranking) at least one alternative in the RUBIS choice (RC).

$\mathcal{P}_2$: Minimal size:
The RC must be as limited in cardinality as possible.

$\mathcal{P}_3$: Stable and efficient:
The RC must not contain a self-contained sub-RC.

$\mathcal{P}_4$: Effectively better (resp. worse):
The RC must not be ambiguous in the sense that it is not both a best choice as well as a worst choice recommendation.

$\mathcal{P}_5$: Maximally significant:
The RC is, of all potential best (resp. worst) choices, the one that is most significantly supported by the marginal "*at least as good as*" relations.

Qualifications of a choice in X

Let S be an r-valued outranking relation defined on X and let Y be a non empty subset of X, called a choice in X.

- Y is called outranking (resp. outranked) iff for all non retained alternative x there exists an alternative y retained such that $r(y\,S\,x) > 0.0$ (resp. $r(x\,S\,y) > 0.0$).
- Y is called independent iff for all $x \neq y$ in Y, we observe $r(x\,S\,y) \leqslant 0.0$.
- Y is an outranking kernel (resp. outranked kernel) iff Y is an outranking (resp. outranked) and independent choice.
- Y is an outranking (resp. outranked) hyper-kernel iff Y is an outranking (resp. outranked) choice containing chordless circuits of odd order $p \geqslant 1$.

Translating the pragmatic RUBIS principles in terms of choice qualifications

$\mathcal{P}_1$: Elimination for well motivated reasons.
The RC is an outranking choice (resp. outranked choice).

$\mathcal{P}_{2+3}$: Minimal and stable choice.
The RC is a hyper-kernel.

$\mathcal{P}_4$: Effectivity.
The RC is a choice which is strictly more outranking than outranked (resp. strictly more outranked than outranking) .

$\mathcal{P}_5$: Maximal significance.
The RC is the most determined one in the set of potential outranking (resp. outranked) hyper-kernels observed in a given r-valued outranking relation.

Properties of the RUBIS choice

Property (decisiveness)

Every r-valued (strict) outranking relation admits at least one outranking and one outranked hyper-kernel.

Definition

Let S and S' be two r-valued outranking relations defined on X.

1. We say that S' upgrades action $x \in X$, denoted $S^{x\uparrow}$, if $r(x\,S'\,y) \geqslant r(x\,S\,y)$, and $r(y\,S'\,x) \leqslant r(y\,S\,x)$, and $r(y\,S'\,z) \;=\; r(y\,S\,z)$ for all $y, z \in X - \{x\}$.
2. We say that S' downgrades action $x \in X$, denoted $S^{x\downarrow}$, if $r(y\,S'\,x) \geqslant r(y\,S\,x)$, and $r(x\,S'\,y) \leqslant r(x\,S\,y)$, and $r(y\,S'\,z) \;=\; r(y\,S\,z)$ for all $y, z \in X - \{x\}$.

Properties of the RUBIS choice

Let A be a subset of X. Let $RBC(S_{|A})$ (resp.$RBC(S'_{|A})$) be the RUBIS best choice wrt to S (resp. S') restricted to A and let $RWC(S_{|A})$ (resp.$RWC(S'_{|A})$) be the RUBIS worst choice wrt to S (resp. S') restricted to A.

Property

1. $S_{|A} = S'_{|A} \;\Rightarrow\; RBC(S_{|A}) = RBC(S'_{|A})$ *(RBC local),*
2. $S_{|A} = S'_{|A} \;\Rightarrow\; RWC(S_{|A}) = RWC(S'_{|A})$ *(RWC local),*
3. $x \in RBC(S_{|A}) \;\Rightarrow\; x \in RBC(S^{x\uparrow}_{|A})$ *(RBC weakly monotonic),*
4. $x \in RWC(S_{|A}) \;\Rightarrow\; x \in RWC(S^{x\downarrow}_{|A})$ *(RWC weakly monotonic).*
5. *The RUBIS choice does not satisfy the Super Set Property (SSP)!*

Ranking-by-Choosing Algorithm

1. Let X_1 be the set X of potential decision actions we wish to rank on the basis of a given outranking relation S.
2. While the remaining set X_i $(i = 1, 2, \ldots)$ of decision actions to be ranked is not empty, we extract from X_i the best (B_i), respectively worst (W_i), RUBIS choice recommendation and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.
3. Both independent iterations determine, hence, two – usually slightly different – opposite weak orderings on X: a ranking-y-best-choosing – and a ranking-by-worst-choosing order.
4. We fuse both rankings, the first and the converse of the second, with the help of the epistemic conjunction operator ($\circledcirc$) to make apparent a weakly complete ranking relation $\succsim_S$ on X. We denote $\succ_S$ the codual of $\succsim_S$.

Transitive S-closure

Definition

We call a ranking procedure transitive if the ranking procedure renders a (partial) strict ordering $\succ_S$ on X with a given r-valued outranking relation S such that for all $x, y, z \in X$: $r(x \succ_S y) > 0$ and $r(y \succ_S z) > 0$ imply $r(x \succ_S z) > 0$.

Property

Both the RUBIS ranking-by-best-choosing, as well as the RUBIS ranking-by-worst-choosing procedures, are transitive ranking procedures.

Corollary

i) The fusion of the ranking by RUBIS best choice and the converse of the ranking by RUBIS worst choice of a given r-valued outranking relation S is a transitive ranking procedure.
ii) The RUBIS ranking-by-choosing represents a transitive closure of the codual of S.

Weak monotinicity

Definition

We call a ranking procedure weakly monotonic if for all $x, y \in X$:
$(x \succ_S y) \;\Rightarrow\; (x \succ_{S^{x\uparrow}} y)$ and $(y \succ_S x) \;\Rightarrow\; (y \succ_{S^{x\downarrow}} x)$,

Property

The ranking by RUBIS best choice and the ranking by RUBIS worst choice are, both, weakly monotonic ranking procedures.

Corollary

The ranking-by-choosing, resulting from the fusion of the ranking by RUBIS best choice and the converse of the ranking by RUBIS worst choice, is hence a weakly monotonic procedure.

Condorcet consistency

Definition

We call a ranking procedure Condorcet-consistent if the ranking procedure renders the same linear (resp. weak) order $\succ_S$ on X which is, the case given, modelled by the strict majority cut of the codual of a given r-valued relation.

Property

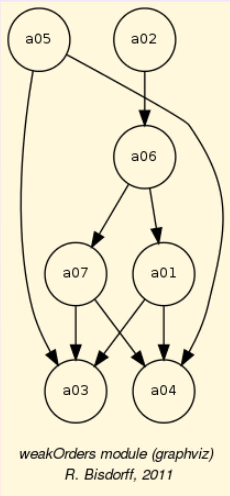
Both the RUBIS ranking-by-best-choosing, as well as the RUBIS ranking-by-worst-choosing procedures, are Condorcet consistent.

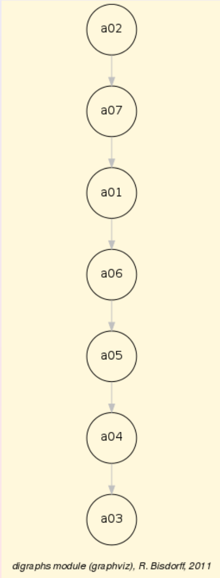
Corollary

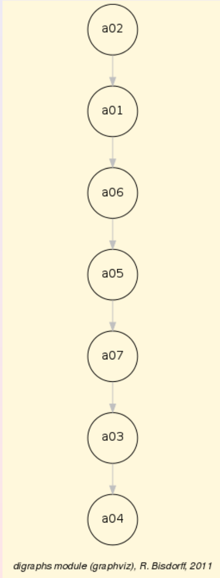
The fusion of the ranking by RUBIS best choice and the converse of the ranking by RUBIS worst choice of a given r-valued outranking relation S is, hence, also Condorcet consistent.

Introductory example

Comparing ranking-by-choosing result with Kohler's and Tideman's:







0.88% 0.90% 0.87%

Sample performance tableau

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Quality of ranking result

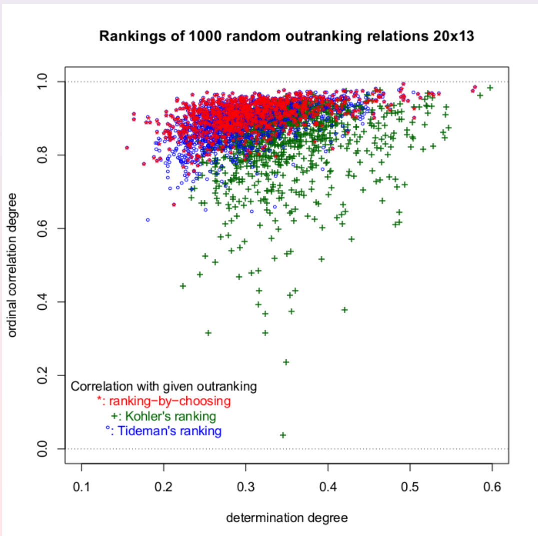
Comparing rankings of a sample of 1000 random r-valued outranking relations defined on 20 actions and evaluated on 13 criteria obtained with RUBIS **ranking-by-choosing**, **Kohler's**, and **Tideman's** (ranked pairs) procedure.

Mean extended Kendall τ correlations with r-valued outranking relation:

Ranking-by-choosing: **+.906**

Tideman's ranking: **+.875**

Kohler's ranking: **+.835**



Quality of ranking-by-choosing result

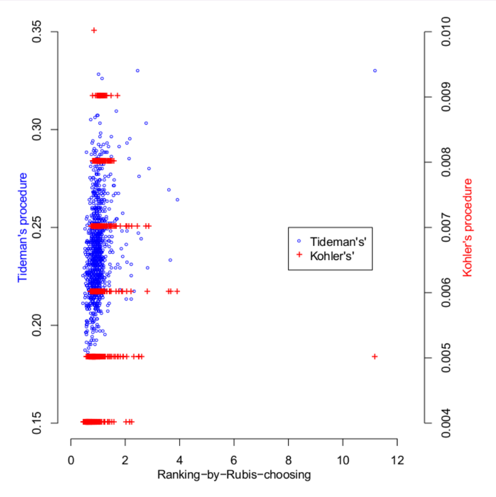
r-valued determination of ranking result:

- Mean outranking significance: 0.351 (67.5% of total criteria support),
- Mean Ranking-by-choosing significance: 0.268 (63.4% of total criteria support),
- Mean covered part of significance: $0.268/0.351 =$ **76%**.

Scalability of ranking procedures

Ranking execution times (in sec.) for 1000 random 20x13 outrankings:

- **Kohler**'s procedure on the right y-axis (less than **1/100** sec.),
- **Tideman**'s procedure on the left y-axis (less than **1/3** sec.),
- the RUBIS **ranking-by-choosing** procedure on the x-axis (mostly less than **2** sec.). But, heavy right tail (up to **11** sec. !).



Practical application

- Spiegel (DE) On-line Students' Survey (2004) about the quality of 41 German universities in 15 academic disciplines;
- XMCDA 2.0 encoding of performace tableau;
- Ranking-by-choosing result.

Bibliography

[1] D. Bouyssou, *Monotonicity of 'ranking by choosing'; A progress report*. Social Choice Welfare (2004) 23: 249-273.

[2] R. Bisdorff, M. Pirlot and M. Roubens, *Choices and kernels from bipolar valued digraphs*. European Journal of Operational Research, 175 (2006) 155-170.

[3] R. Bisdorff, P. Meyer and M. Roubens, RUBIS *: a bipolar-valued outranking method for the choice problem*. 4OR, A Quarterly Journal of Operations Research, Springer-Verlag, Volume 6 Number 2 (2008) 143-165.

[4] R. Bisdorff, *On measuring and testing the ordinal correlation between bipolar outranking relations*. In Proceedings of DA2PL'2012 - *From Multiple Criteria Decision Aid to Preference Learning*, University of Mons (2012) 91-100.

[5] R. Bisdorff, *On polarizing outranking relations with large performance differences*. Journal of Multi-Criteria Decision Analysis, Wiley, Number 20 (2013) DOI: 10.1002/mcda.1472 3-12.