# Ranking big multicriteria performance tableaux

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## Motivation: showing a performance tableau

Consider a performance table showing the service quality of 12 commercial cloud providers measured by an external auditor on 14 incommensurable performance criteria.

**Performance table auditor2_1**

| criterion | upT | dwT | ouT | LB | MTBF | Rcv | Lat | RspT | Thrpt | stoC | snpC | auT | enC | auD |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Amz | 2 | 2 | 2 | 4 | 3 | 3 | NA | 3 | NA | 4 | NA | 4 | 4 | 4 |
| Cen | 4 | 4 | 0 | 4 | 4 | 4 | NA | 2 | NA | 3 | NA | 4 | 4 | 4 |
| Cit | 2 | 4 | 2 | 4 | 3 | 4 | NA | 2 | NA | 3 | 4 | 4 | 4 | 4 |
| Dig | 2 | 1 | 4 | 4 | 3 | 3 | NA | 2 | NA | 3 | NA | 4 | 4 | 4 |
| Ela | 4 | 4 | 0 | 4 | 4 | 4 | NA | 4 | NA | 3 | 4 | 4 | 4 | 4 |
| GMO | 1 | 3 | 4 | 4 | 3 | 2 | NA | 4 | NA | 3 | NA | 4 | 4 | 4 |
| Ggl | 4 | 2 | 1 | 4 | 2 | 3 | NA | 2 | NA | 4 | 4 | 4 | 4 | 4 |
| HP | 3 | 3 | 2 | 4 | 4 | 3 | NA | 4 | NA | 3 | 4 | 4 | 4 | 4 |
| Lux | 2 | 2 | 2 | 4 | 3 | 3 | NA | 2 | NA | 2 | NA | 4 | 4 | 4 |
| MS | 4 | 4 | 0 | 4 | 4 | 4 | NA | 4 | NA | 4 | NA | 4 | 4 | 4 |
| Rsp | NA | NA | NA | 4 | NA | 3 | NA | NA | NA | 3 | 4 | 4 | 4 | 4 |
| Sig | 4 | 4 | 0 | 4 | 4 | 4 | NA | 3 | NA | 3 | 4 | 4 | 4 | 4 |

**Legend:** 0 = '*very weak*', 1 = '*weak*', 2 = '*fair*', 3 = '*good*', 4 = '*very good*','*NA*' = missing data; '*green*' and '*red*' mark the **best**, respectively the **worst**, performances on each criterion.





## Motivation: showing an ordered heat map

The same performance tableau may be optimistically colored with the highest 7-tiles class of the marginal performances and presented like a heat map,

**Heatmap of Performance Tableau 'auditor2_1'**

| criteria | stoC | snpC | upT | dwT | ouT | LB | MTBF | Rcv | Lat | RspT | Thrpt | auT | enC | auD |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| weights | 3.00 | 3.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 1.00 | 1.00 | 1.00 |
| MS | 4 | NA | 4 | 4 | 0 | 4 | 4 | 4 | NA | 4 | NA | 4 | 4 | 4 |
| Ela | 3 | 4 | 4 | 4 | 0 | 4 | 4 | 4 | NA | 4 | NA | 4 | 4 | 4 |
| Sig | 3 | 4 | 4 | 4 | 0 | 4 | 4 | 4 | NA | 3 | NA | 4 | 4 | 4 |
| Cen | 3 | NA | 4 | 4 | 0 | 4 | 4 | 4 | NA | 2 | NA | 4 | 4 | 4 |
| HP | 3 | 4 | 3 | 3 | 2 | 4 | 4 | 3 | NA | 4 | NA | 4 | 4 | 4 |
| Cit | 3 | 4 | 2 | 4 | 2 | 4 | 3 | 4 | NA | 2 | NA | 4 | 4 | 4 |
| Ggl | 4 | 4 | 4 | 2 | 1 | 4 | 2 | 3 | NA | 2 | NA | 4 | 4 | 4 |
| GMO | 3 | NA | 1 | 3 | 4 | 4 | 3 | 2 | NA | 4 | NA | 4 | 4 | 4 |
| Rsp | 3 | 4 | NA | NA | NA | 4 | NA | 3 | NA | NA | NA | 4 | 4 | 4 |
| Amz | 4 | NA | 2 | 2 | 2 | 4 | 3 | 3 | NA | 3 | NA | 4 | 4 | 4 |
| Dig | 3 | NA | 2 | 1 | 4 | 4 | 3 | 3 | NA | 2 | NA | 4 | 4 | 4 |
| Lux | 2 | NA | 2 | 2 | 2 | 4 | 3 | 3 | NA | 2 | NA | 4 | 4 | 4 |

*Color legend:*

| quantile | 0.14% | 0.29% | 0.43% | 0.57% | 0.71% | 0.86% | 1.00% |
|---|---|---|---|---|---|---|---|

eventually linearly ordered, following for instance the Copeland ranking rule, from the best to the worst performing alternatives (ties are lexicographically resolved).





## How to handle big performance tableaux ?

- The Copeland ranking rule is based on crisp net flows requiring the in- and out-degree of each node in the outranking digraph;
- When the order $n$ of the outranking digraph becomes big (several thousand), this requires handling a huge set of $n^2$ pairwise outranking situations;
- We shall present hereafter a sparse model of the outranking digraph, where we only keep a linearly ordered list of quantiles equivalence classes with local outranking content.

# Content







# Performance quantiles

- Let $X$ be the set of $n$ potential decision alternatives evaluated on a single real performance criteria.
- We denote $x, y, \ldots$ the performances observed of the potential decision actions in $X$.
- We call quantile $q(p)$ the performance such that $p\%$ of the observed $n$ performances in $X$ are less or equal to $q(p)$.
- The quantile $q(p)$ is estimated by linear interpolation from the cumulative distribution of the performances in $X$.





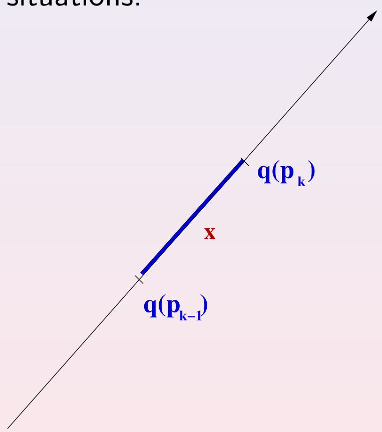
# Performance quantile classes

- We consider a series: $p_k = k/q$ for $k = 0, \ldots q$ of $q+1$ equally spaced quantiles like
  - quartiles: $0, .25, .5, .75, 1$,
  - quintiles: $0, .2, .4, .6, .8, 1$,
  - deciles: $0, .1, .2, \ldots, .9, 1$, etc
- The upper-closed $q^k$ class corresponds to the interval $]q(p_{k-1}); q(p_k)]$, for $k = 2, \ldots, q$, where $q(p_q) = \max_X x$ and the first class gathers all data below $p_1$: $]-\infty; q(p_1)]$.
- The lower-closed $q_k$ class corresponds to the interval $[q(p_{k-1}); q(p_k)[$,for $k = 1, \ldots, q-1$, where $q(p_0) = \min_X x$ and the last class gathers all data above $q(p_{q-1})$: $[q(p_{q-1}), +\infty[$.
- We call $q$-tiles a complete series of $k = 1, \ldots, q$ upper-closed $q^k$, resp. lower-closed $q_k$, quantile classes.



# $q$-tiles sorting on a single criteria

If $x$ is a measured performance, we may distinguish three sorting situations:




1. $x \leqslant q(p_{k-1})$ and $x < q(p_k)$
   The performance $x$ is lower than the $q^k$ class;
2. $x > q(p_{k-1})$ and $x \leqslant q(p_k)$
   The performance $x$ belongs to the $q^k$ class;
3. ($x > q(p_{k-1})$ and) $x > q(p_k)$
   The performance $x$ is higher than the $p^k$ class.

If the relation $<$ is the dual of $\geqslant$, it will be sufficient to check that both, $q(p_{k-1}) \not\geqslant x$, as well as $q(p_k) \geqslant x$, are verified for $x$ to be a member of the $k$-th $q$-tiles class.

## Multiple criteria extension

- $A = \{x, y, z, ...\}$ is a finite set of $n$ objects to be sorted.
- $F = \{1, ..., m\}$ is a finite and coherent family of $m$ performance criteria.
- For each criterion $j$ in $F$, the objects are evaluated on a real performance scale $[0; M_j]$,

  supporting an indifference threshold $ind_j$

  and a preference threshold $pr_j$ such that $0 \leqslant ind_j < pr_j \leqslant M_j$.
- The performance of object $x$ on criterion $j$ is denoted $x_j$.
- Each criterion $j$ in $F$ carries a rational significance $w_j$ such that $0 < w_j < 1.0$ and $\sum_{j \in F} w_j = 1.0$.



## Performing marginally *at least as good as*

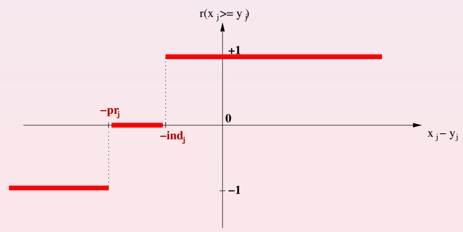
Each criterion $j$ is characterizing a double threshold order $\geqslant_i$ on $A$ in the following way:

$$r(x \geqslant_j y) = \begin{cases} +1 & \text{if} \quad x_j - y_j \geqslant -ind_j \\ -1 & \text{if} \quad x_j - y_j \leqslant -pr_j \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

+1 signifies $x$ is *performing at least as good as* $y$ on criterion $j$,

−1 signifies that $x$ is *not performing at least as good as* $y$ on criterion $j$.

0 signifies that it is *unclear* whether, on criterion $j$, $x$ is performing at least as good as $y$.






## Performing globally *at least as good as*

Each criterion $j$ contributes the significance $w_j$ of his "*at least as good as*" characterization $r(\geqslant_j)$ to the global characterization $r(\geqslant)$ in the following way:

$$\boxed{r(x \geqslant y) \;=\; \sum_{j \in F} \left[\, w_j \cdot r(x \geqslant_j y) \,\right]} \quad (2)$$

$r > 0$ signifies $x$ is *globally performing at least as good as* $y$,

$r < 0$ signifies that $x$ is *not globally performing at least as good as* $y$,

$r = 0$ signifies that it is *unclear* whether $x$ is globally performing at least as good as $y$.





## The bipolar outranking relation $\succsim$

From an epistemic point of view, we say that:

1. object $x$ *outranks* object $y$, denoted $(x \succsim y)$, if
   1.1 a significant majority of criteria validates a global outranking situation between $x$ and $y$, i.e. $(x \geqslant y)$ and
   1.2 no veto $(x \not\lll y)$ is observed on a discordant criterion,
2. object $x$ *does not outrank* object $y$, denoted $(x \not\succsim y)$, if
   2.1 a significant majority of criteria invalidates a global outranking situation between $x$ and $y$, i.e. $(x \not\geqslant y)$ and
   2.2 no counter-veto $(x \not\ggg y)$ observed on a concordant criterion.

## Polarising the global "*at least as good as*" characteristic

The valued bipolar outranking characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = \begin{cases} 0, & \text{if } \big[\exists j \in F : r(x \lll_j y)\big] \wedge \big[\exists k \in F : r(x \ggg_k y)\big] \\ \big[\, r(x \geqslant y) \circledvee -r(x \lll y) \,\big] & \text{, otherwise.} \end{cases}$$

And in particular,

- $r(x \succsim y) = r(x \geqslant y)$ if no very large positive or negative performance differences are observed,
- $r(x \succsim y) = 1$ if $r(x \geqslant y) \geqslant 0$ and $r(x \ggg y) = 1$,
- $r(x \succsim y) = -1$ if $r(x \geqslant y) \leqslant 0$ and $r(x \lll y) = 1$,





## $q$-tiles sorting with bipolar outrankings

### Proposition

*The bipolar characteristic of $x$ belonging to upper-closed $q$-tiles class $q^k$, resp. lower-closed class $q_k$, may hence, in a multiple criteria outranking approach, be assessed as follows:*

$$r(x \in q^k) = \min\big[\, -r\big(\mathbf{q}(p_{k-1}) \succsim x\big),\; r\big(\mathbf{q}(p_k) \succsim x\big)\,\big]$$
$$r(x \in q_k) = \min\big[\, r\big(x \succsim \mathbf{q}(p_{k-1})\big),\; -r\big(x \succsim \mathbf{q}(p_k)\big)\,\big]$$

### Proof.

The bipolar outranking relation $\succsim$, being weakly complete, verifies the coduality principle (Bisdorff 2013). The dual $(\not\succsim)$ of $\succsim$ is, hence, identical to the strict converse outranking $\precnsim$ relation. □





## The multicriteria (upper-closed) $q$-tiles sorting algorithm

1. **Input**: a set $X$ of $n$ objects with a performance table on a family of $m$ criteria and a set $\mathcal{Q}$ of $k = 1, .., q$ empty $q$-tiles equivalence classes.
2. **For each** object $x \in X$ **and each** $q$-tiles class $q^k \in \mathcal{Q}$
   - 2.1 $r(x \in q^k) \quad \leftarrow \quad \min\big(-r(\mathbf{q}(p_{k-1}) \succsim x), r(\mathbf{q}(p_k) \succsim x)\big)$
   - 2.2 if $r(x \in q^k) \geqslant 0$ :
     **add** $x$ to $q$-tiles class $q^k$
3. **Output**: $\mathcal{Q}$

### Comment

1. *The complexity of the $q$-tiles sorting algorithm is $\mathcal{O}(nmq)$; linear in the number of decision actions ($n$), criteria ($m$) and quantile classes ($q$).*
2. *As $\mathcal{Q}$ represents a partition of the criteria measurement scales, i.e. the upper limits of the preceding category correspond to the lower limits of the succeeding ones, there is a potential for run time optimization.*

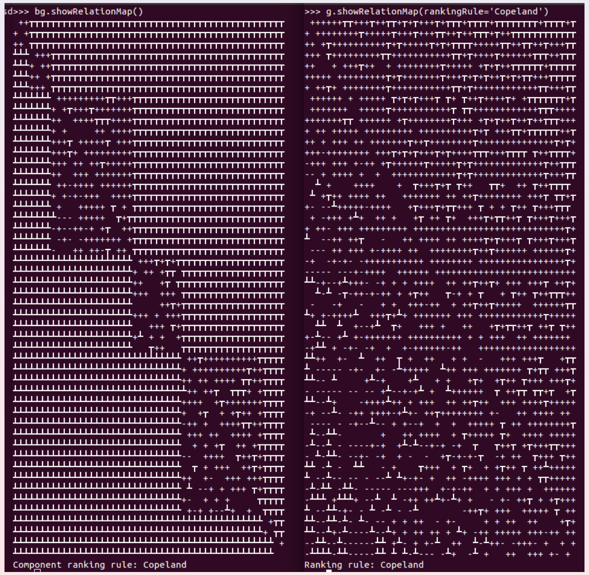




## Example of sparse outranking Digraph

```
>>> from bigOutrankingDigraphs import *
>>> t = RandomPerformanceTableau(numberOfActions=50,seed=5)
>>> bg = BigOutrankingDigraphMP(t,quantiles=5)
>>> bg.showDecomposition()
*--- quantiles decomposition in decreasing order---*
c1. [0.60-0.80[ : ['a22','a24','a32']
c2. [0.40-0.80[ : ['a16', 'a28','a31','a40']
c3. [0.40-0.60[ : ['a01','a02','a05','a06','a10',
                   'a13','a15','a25','a27','a35',
                   'a36','a37','a39','a41','a48']
c4. [0.20-0.60[ : ['a09','a14','a18','a20','a26',
                   'a38','a43','a45','a49']
c5. [0.20-0.40[ : ['a03','a04','a07','a08','a11',
                   'a12','a17','a21','a29','a30',
                   'a33','a34','a42','a44','a47']
c6. [0.00-0.40[ : ['a46','a50']
c7. [0.00-0.20[ : ['a19','a23']
```

## Sparse versus standard outranking digraph of order 50




**Symbol legend**

- ⊤ certainly valid
- $+$ more or less valid
- ' ' indeterminate
- - more or less invalid
- ⊥ certainly invalid

**Sparse digraph** *bg*:
# Actions : 50
# Criteria : 7
Sorted by : 5-Tiling
Ranking rule : Copeland
# Components : 7
Minimal order : 1
Maximal order : 15
Average order : 7.1
fill rate : 20.980%
correlation : +0.7563



## Content







## Properties of $q$-tiles sorting result

1. *Coherence*: Each object is always sorted into a non-empty subset of adjacent $q$-tiles classes.
2. *Uniqueness*: If the $q$-tiles classes represent a discriminated partition of the measurement scales on each criterion and $r \neq 0$, then every object is sorted into exactly one $q$-tiles class.
3. *Independence*: The sorting result for object $x$, is independent of the other object's sorting results.

### Comment

*The independence property gives us access to efficient parallel processing of class membership characteristics $r(x \in q^k)$ for all $x \in X$ and $q^k$ in $\mathcal{Q}$.*





## Ordering the $q$-tiles sorting result

The $q$-tiles sorting result leaves us with a more or less refined partition of the set $X$ of $n$ potential decision actions. For linearly ranking from best to worst the resulting parts of the $q$-tiles partition we may apply three strategies:

1. Optimistic: In decreasing lexicographic order of the upper and lower quantile class limits;
2. Pessimistic: In decreasing lexicographic order of the lower and upper quantile class limits;
3. Average: In decreasing numeric order of the average of the lower and upper quantile limits.

## $q$-tiles ranking algorithm

1. **Input**: the outranking digraph $\mathcal{G}(X, \succsim)$, a partition $P_q$ of $k$ linearly ordered decreasing parts of $X$ obtained by the $q$-sorting algorithm, and an empty list $\mathcal{R}$.
2. **For each** quantile class $q^k \in P_q$:
   **if** $\#(q^k) > 1$:
   $R_k \leftarrow$ **locally rank** $q^k$ in $\mathcal{G}_{|q^k}$
   (if ties, render alphabetic order of action keys)
   **else**: $R_k \leftarrow q^k$
   **append** $R_k$ to $\mathcal{R}$
3. **Output**: $\mathcal{R}$





## $q$-tiles ranking algorithm – Comments

1. The complexity of the $q$-tiles ranking algorithm is linear in the number $k$ of components resulting from a $q$-tiles sorting which contain more than one action.
2. Three local ranking rules are available – *Copeland's*, *Net-flows'* and *Kohler's rule* – of complexity $\mathcal{O}(\#(q^k)^2)$ on each restricted outranking digraph $\mathcal{G}_{|q^k}$.
3. In case of local ties (very similar evaluations for instance), the **local ranking** procedure will render these actions in increasing alphabetic ordering of the action keys.





## Content

1. Sparse model of outranking digraph
   Single criteria quantiles sorting
   Multiple criteria quantiles sorting
2. Ranking a $q$-tiled performance tableau
   Properties of the $q$-tiles sorting
   Ordering the $q$-tiles sorting result
   $q$-tiles ranking algorithm
3. HPC-ranking a big performance tableau
   Multithreading the sorting&ranking procedure
   Profiling the HPC sorting&ranking procedure





## Multithreading the $q$-tiles sorting & ranking procedure

1. Following from the independence property of the **$q$-tiles sorting** of each action into each $q$-tiles class, the $q$-sorting algorithm may be safely split into as much threads as are multiple processing cores available in parallel.
2. Furthermore, the **ranking** procedure being local to each diagonal component, these procedures may hence be safely processed in parallel threads on each restricted outranking digraph $\mathcal{G}_{|q^k}$.

# Generic algorithm design for parallel processing

```
from multiprocessing import Process, active_children
    class myThread(Process):
        def __init__(self, threadID, ...)
                    Process.__init__(self)
                    self.threadID = threadID
                    ...
        def run(self):
            ... task description
            ...

nbrOfJobs = ...
for job in range(nbrOfJobs):
    ... pre-threading tasks per job
    print('iteration = ',job+1,end=" ")
    splitThread = myThread(job, ...)
    splitThread.start()
while active_children() != []:
    pass
print('Exiting computing threads')
for job in range(nbrOfJobs):
    ... post-threading tasks per job
```





# Choosing the right granularity ?

Is it more efficient:

- to run many simple jobs in parallel ?
- to run in parallel a small number of complex jobs ?
- to align the number of parallel jobs to the number of available cores ?
- to start more parallel jobs than available cores ?





# HPC performance measurements

| digraph | standard model | | | sparse model | | |
|---|---|---|---|---|---|---|
| order | #c. | $t_g$ sec. | $\tau_g$ | #c. | $t_{bg}$ sec. | $\tau_{bg}$ |
| 500 | 50 | 7 | +0.88 | 160 | 2 | +0.83 |
| 1 000 | 50 | 27 | +0.88 | 160 | 3 | +0.83 |
| 2 000 | 50 | 108 | +0.88 | 160 | 7 | +0.83 |
| 2 500 | 50 | 160 | +0.88 | 160 | 9 | +0.83 |
| 10 000 | | | | 160 | 49 | |
| 15 000 | | | | 160 | 72 | |
| 50 000 | | | | 119 | 425 | |
| 100 000 | | | | 119 | 854 | |
| 200 000 | | | | 119 | 2232 | |
| 250 000 | | | | 119 | 3417 | |

**Legend:**

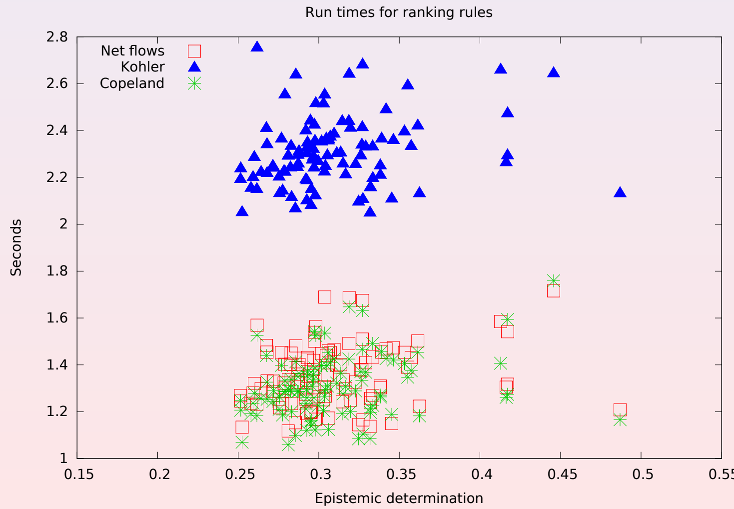
- #c. = number of cores;
- $g$: standard outranking digraph, $bg$: the sparse outranking digraph;
- $t_g$, resp. $t_{bg}$, are the corresponding constructor run times;
- $\tau_g$, resp. $\tau_{bg}$ are the ordinal correlation of the Copeland ordering with the given outranking relation.





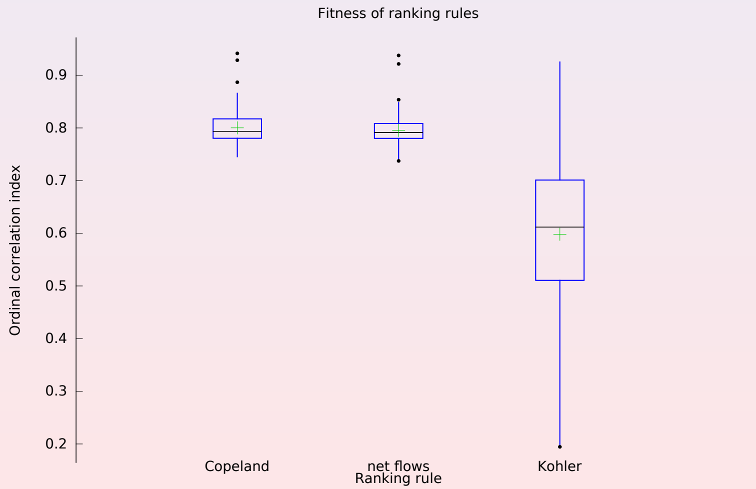
# Choosing a ranking rule – run time statistics

Sample of 100 random outranking graphs of order 250



## Choosing a ranking rule – fitness of ranking rule

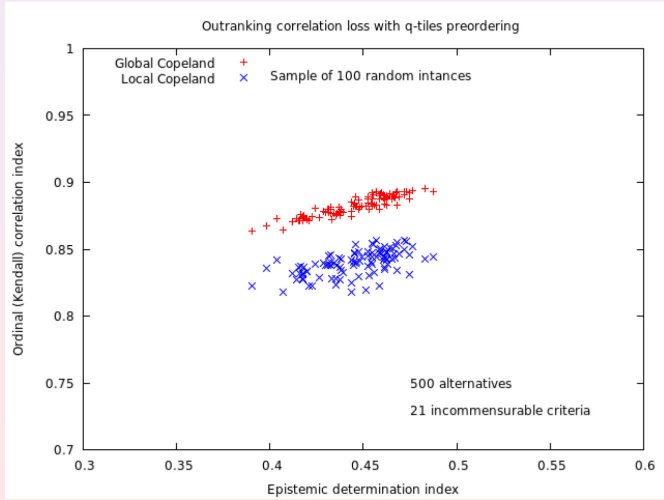
Sample of 100 random outranking graphs of order 250







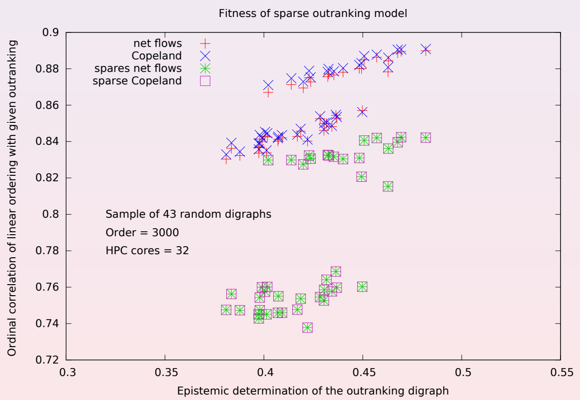

## Standard versus 50-tiled sparse outranking digraphs






## Profiling the local ranking procedure

It is opportune to use Copeland's rule for ranking form the standard outranking digraph, whereas both, Net Flows and Copeland's ranking rule, are equally efficient on the sparse outranking digraph.




The quality of the sparse model based linear ordering is depending on the alignment of the given outranking digraph, but not on its actual order.



## Concluding ...

- We implement a sparse outranking digraph model coupled with a linearly ordering algorithm based on quantiles-sorting & local-ranking procedures;
- Global ranking result fits apparently well with the given outranking relation;
- Independent sorting and local ranking procedures allow effective multiprocessing strategies;
- Efficient scalability allows hence the ranking of very large sets of potential decision actions (thousands of nodes) graded on multiple incommensurable criteria;
- Good perspectives for further optimization and ad-hoc fine-tuning.